Math 544, Exam 2, Spring 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **7** problems on **two sides**. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. (8 points) **Let v_1 and v_2 be fixed vectors in $\mathbb{R}^n$, for some n, and let**

$$V = \{v \in \mathbb{R}^n \mid v_1^{\mathrm{T}} v = 0 \textbf{ and } v_2^{\mathrm{T}} v = 0\}.$$

Is V a vector space? If yes, prove the statement. If no, show an example in which one of the rules of vector spaces is violated.

YES, V is a vector space.

The zero vector is in V because $v_1^{\mathrm{T}} v = 0$ and $v_2^{\mathrm{T}} v = 0$.

The set V is closed under addition. If v and v' are in V, then $v + v'$ is in V because for i equal to 1 and 2,

$$v_i^{\mathrm{T}}(v + v') = v_i^{\mathrm{T}}(v) + v_i^{\mathrm{T}}(v') = 0 + 0 = 0.$$

The first equality holds because matrix multiplicxation distributes over addition. The second equality holds because v and v' are in V.

The set V is closed under scalar multiplication. If v is in V and λ is a scalar, then λv is in V because $v_i^{\mathrm{T}}(\lambda v) = \lambda v_i^{\mathrm{T}}(v) = \lambda 0 = 0$.

The set V satisfies all three rules for being a vector space; thus, V is a vector space.

2. (7 points) **Let**

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, ab = 0 \right\}.$$

Is V a vector space? If yes, prove the statement. If no, show that one of the rules of vector spaces is violated.

NO, V is not a vector space because V is not closed under addition. Indeed

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are in V because $1 \cdot 0 = 0$; but $v_1 + v_2$ is not in V because $1 \cdot 1 \neq 0$.

3. (7 points) **Let A and B be 2×2 matrices with $A^2 = B^2$. Does B have to equal A or $-A$? If yes, prove the statement. If no, show an example.**

NO, many matrices square to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, for example $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$ square to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all real numbers a.

4. (7 points) **Consider the vectors**

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \textbf{and} \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Are the vectors v_1, v_2, v_3 linearly independent? Demonstrate that your answer is correct. (It might be useful to notice that $v_i^{\mathrm{T}} v_j = 0$ whenever $i \neq j$.)

These vectors are linearly independent. Indeed, if $\sum a_j v_j = 0$, then we multiply both sides of the equation by v_1^{T} to learn that $4a_1 = 0$ (hence $a_1 = 0$). Multiply both sides of the equation by v_2^{T} to learn that $4a_2 = 0$ (hence $a_2 = 0$). Multiply both sides of the equation by v_3^{T} to learn that $2a_3 = 0$ (hence $a_3 = 0$). At any rate if $\sum a_j v_j = 0$, then all three a's have to be zero. The three v's are linearly independent.

5. (7 points) **Let A and B be $n \times n$ matrices with $AB = I$. Does BA have to equal I? If yes, prove the statement. If no, show an example.**

YES. The equation $AB = I$ guarantees that B is non-singular. (If v is a vector and $Bv = 0$, then $ABv = 0$; but $AB = I$ and $Iv = v$; so $v = 0$.) The non-singular matrix theorem now guarentees that B is invertible. Multiple both sides of $AB = I$ on the right by B^{-1} to learn $ABB^{-1} = B^{-1}$. It follows that $A = B^{-1}$. Now multiply both sides by B to learn $BA = BB^{-1}$. Of course, $BB^{-1} = I$. We conclude that $BA = I$.

6. (7 points) **Let A be an $m \times n$ matrix and B be an $n \times m$ matrix with $AB = I$. Does BA have to equal I? If yes, prove the statement. If no, show an example.**

NO. Let $A = [0 \quad 1]$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We see that $AB = [1]$, the 1×1 identity matrix; but $BA = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, which is not the identity matrix.

7. **Consider the system of equations** $Ax = b$ **where** $A = \begin{bmatrix} a & -3a+3 \\ 1 & a-1 \end{bmatrix}$**,** $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$**, and** $b = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$**.**
   (a) **For which values of** a **does the system of equations have no solution?**
   (b) **For which values of** a **does the system of equations have exactly one solution?**
   (c) **For which values of** a **does the system of equations have more than one solution?**

If M is the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det M \neq 0$, then M is invertible with inverse $M^{-1} = \frac{1}{\det M}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. (Recall that $\det M = ad - bc$.) In this case every system of equations $Mx = b$ has a unique solution by the non-singular matrix theorem. For our matrix, $\det A = a(a-1)-(-3a+3) = a^2+2a-3 = (a-1)(a+3)$. So, if a is not 1 or -3, then $Ax = b$ has a unique solution.

If $a = 1$, the system of equations is $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$. Of course, the two equations represent parallel lines and there is no solution to the system of equations.

If $a = -3$, the system of equations is $\begin{bmatrix} -3 & 12 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$. The top equation is -3 times the bottom equation. Every point on the line $x_1 - 4x_2 = 3$ satisfies both equations.

$$(a) \quad \boxed{\text{If } a = 1 \text{ then } Ax = b \text{ has no solution.}}$$

$$(b) \quad \boxed{\text{If } a \text{ is not } 1 \text{ or } -3\text{, then } Ax = b \text{ has a unique solution.}}$$

$$(c) \quad \boxed{\text{If } a = -3 \text{ then } Ax = b \text{ has many solutions.}}$$