Math 544, Exam 2, Summer 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **8** problems. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. (6 points) **Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.**

The $n \times n$ matrix A is <u>non-singular</u> if the only vector x in $\mathbb{R}^n$ with $Ax = 0$ is $x = 0$.

2. (7 points) **Let A be an $n \times n$ matrix. List three statements that are equivalent to the statement "A is non-singular". Please do not repeat your answer to problem 1 and only use statements that we have studied thoroughly.**

   1. The columns of A are linearly independent.
   2. The system of equations $Ax = b$ has a unique solution for every b in $\mathbb{R}^n$.
   3. The matrix A is invertible.

3. (7 points) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum\limits_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

4. (6 points) **Suppose that v_1, v_2, and v_3 are linearly independent vectors in $\mathbb{R}^n$ and M is an $n \times n$ matrix. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

NO! Take $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see that v_1, v_2, v_3 are linearly independent; but Mv_1, Mv_2, Mv_3 are linearly dependent.

5. (6 points) **Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be an $n \times n$ matrix. Suppose the vectors Mv_1, Mv_2, Mv_3 are linearly independent. Do the vectors v_1, v_2, v_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

YES! Let c_1, c_2, and c_3 be numbers with $c_1v_1 + c_2v_2 + c_3v_3 = 0$. Multiply by M and distribute to see that $c_1Mv_1 + c_2Mv_2 + c_3Mv_3 = 0$. The vectors Mv_1, Mv_2, Mv_3 are linearly independent; hence, $c_1 = c_2 = c_3$.

6. (6 points) **Suppose that v_1, v_2, and v_3 are linearly independent vectors in $\mathbb{R}^n$ and M is an invertible $n \times n$ matrix. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

YES! Let c_1, c_2, and c_3 be numbers with $c_1Mv_1+c_2Mv_2+c_3Mv_3 = 0$. Multiply by M^{-1} and distribute to see that $c_1v_1 + c_2v_2 + c_3v_3 = 0$. The vectors v_1, v_2, v_3 are linearly independent; hence, $c_1 = c_2 = c_3$.

7. (6 points) **Let A be the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. What is the inverse of A? Please make sure that your answer is correct.**

The inverse of A is $A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$. We check that $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

8. (6 points) **Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \middle| ab = 0 \right\}$? Is V a vector space? If yes, prove it. If not, illustrate with an example.**

NO! We see that $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are both in V, but $v_1 + v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is not in V.