Math 544, Exam 2, Spring 2011

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **9** problems on **TWO SIDES**. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. (5 points) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum\limits_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

2. (5 points) **Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.**

The square matrix A is non-singular if the **only** column vector x with $Ax = 0$ is $x = 0$.

3. (5 points) **State the Non-Singular Matrix Theorem. Your statement should consist of four equivalent conditions. (I will only accept concepts that we have discussed thoroughly in class.) Include everything that is necessary, but nothing more. Use complete sentences.**

Let A be an $n \times n$ matrix. The following statements are equivalent.

(1) The matrix A is non-singular.
(2) The columns of A are linearly independent.
(3) For every vector b in $\mathbb{R}^n$, there is exactly one vector v with $Av = b$.
(4) The matrix A is invertible.

4. (5 points) **Let U and V be subspaces of $\mathbb{R}^n$. Recall that the** *intersection* **of U and V is the set of all elements of $\mathbb{R}^n$ that are in both U and V. Let W be the intersection of U and V. Is W always a vector space? If "yes", then prove that W is a vector space. If "no", then give an**

example that demonstrates why W is not always a vector space. Use complete sentences. Justify each step.

YES!

The vector 0 is in U (since U is a vector space). The vector 0 is in V (since V is a vector space). Thus, 0 is in W, which is the intersection of U and V.

The set W is closed under addition. Take w_1 and w_2 in W. We have w_1 and w_2 both in U and U is a vector space; thus, $w_1 + w_2$ is in U. We have w_1 and w_2 both in V and V is a vector space; thus, $w_1 + w_2$ is in V. Thus, $w_1 + w_2$ is in W, which is the intersection of U and V.

The set W is closed under scalar multiplication. Take w in W and a real number c. We have w is in U and U is a vector space; thus, cw is in U. We have w is in V and V is a vector space; thus, cw is in V. Thus, cw is in W, which is the intersection of U and V.

We have shown that W contains zero and is closed under addition and scalar multiplication. It follows that W is a vector space.

5. (6 points) **Let U and V be subspaces of $\mathbb{R}^n$. Recall that the *union* of U and V is the set of all elements of $\mathbb{R}^n$ that are in at least one of the sets U or V. Let W be the union of U and V. Is W always a vector space? If "yes", then prove that W is a vector space. If "no", then give an example that demonstrates why W is not always a vector space. Use complete sentences. Justify each step.**

NO!

Let $U = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \middle| x \in \mathbb{R} \right\}$ and $V = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| y \in \mathbb{R} \right\}$. Observe that U and V are both vector spaces. We now show that W, which is the union of U and V, is not a vector space. Observe that $w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in W (since it is in U) and $w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in W (since it is in V) but $w_1 + w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in W. The bottom entry of $w_1 + w_2$ is not zero; so $w_1 + w_2$ is not in U. The top entry of $w_1 + w_2$ is not zero; so $w_1 + w_2$ is not in V. The set W is not closed under addition. The set W is not a vector space.

6. (6 points) **Let M be an $m \times n$ matrix. Let $W = \{v \in \mathbb{R}^n \mid Mv = 0\}$. Is W always a vector space? If "yes", then prove that W is a vector**

space. If "no", then give an example that demonstrates why W is not always a vector space. Use complete sentences. Justify each step.

YES!

The zero vector is in W because $M0 = 0$.

The set W is closed under addition. Take w_1 and w_2 in W. Observe that

$$M(w_1 + w_2) = Mw_1 + Mw_2 = 0 + 0 = 0.$$

The first equality holds because matrix multoiplication distributes across addition. The second equality holds because w_1 and w_2 are in W. At any rate, we have shown that $w_1 + w_2$ is in W.

The set W is closed under scalar multiplication. Take w in W and a number c. Observe that

$$M(cw) = cMw = c(0) = 0.$$

In the first equality, we factor c out of each row of the product $M(cw)$. The second equality holds because w is in W. We see that cw is in W.

We have shown that W contains zero and is closed under addition and scalar multiplication. It follows that W is a vector space.

7. (6 points) **Let A and B be $n \times n$ matrices. How is $(AB)^{\mathrm{T}}$ related to the product of A^{T} and B^{T}? Prove that your answer is correct. For full credit your answer should work for all n. Justify each step. Your notation must make sense. Use complete sentences.**

We prove that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. The matrices $(AB)^{\mathrm{T}}$ and $B^{\mathrm{T}}A^{\mathrm{T}}$ both are $n \times n$ matrices. We prove that the corresponding entries are equal. Let $A_{i,j}$ be the entry of A in row i and column j. The entry in row r column c of $(AB)^{\mathrm{T}}$ is

$$\begin{aligned}
[(AB)^{\mathrm{T}}]_{r,c} = (AB)_{c,r} &= \sum_{j=1}^{n} A_{c,j}B_{j,r} = \sum_{j=1}^{n} B_{j,r}A_{c,j} \\
&= \sum_{j=1}^{n} (B^{\mathrm{T}})_{r,j}(A^{\mathrm{T}})_{j,c} = [(B^{\mathrm{T}})(A^{\mathrm{T}})]_{r,c}
\end{aligned}$$

and this is the entry in row r and column c of $(B^{\mathrm{T}})(A^{\mathrm{T}})$. The first equality is the definition of transpose. The second equality is the definition of matrix product. The symbols $A_{c,j}$ and $B_{j,r}$ represent numbers and numbers commute under multiplication. This explains the third equality. The fourth equlaity is the definition of transpose, again. The fifth equality is the definition of matrix multiplication, again.

8. (6 points) **Let A and B be $n \times n$ matrices with AB non-singular. Prove that A and B are both non-singular. Use complete sentences. Justify each step.**

We first show that B is non-singular. Suppose that v is a vector with $Bv = 0$. Multiplication by A gives $ABv = A0 = 0$. The matrix AB is non-singular and $ABv = 0$. It follows that $v = 0$.

Now we show that A is non-singular. Suppose that v is a vector with $Av = 0$. We saw above that the matrix B is non-singular. It follows from the non-singular matrix theorem that B is invertible. Let B^{-1} be the inverse of B. We have $0 = Av = AB(B^{-1}v)$. The matrix AB is non-singular; so, $B^{-1}v = 0$. Multiply by B to see that $BB^{-1}v = B0 = 0$. Thus, v, which is equal to $BB^{-1}v$, is the zero vector.

9. (6 points) **Let v_1, v_2, v_3 be vectors in $\mathbb{R}^4$ with**
   (a) v_1, v_2 **a linearly independent set of vectors,**
   (b) v_1, v_3 **a linearly independent set of vectors, and**
   (c) v_2, v_3 **a linearly independent set of vectors.**

Does v_1, v_2, v_3 have to be a linearly independent set of vectors ? If yes, prove your answer. If no, give a counterexample. Use complete sentences.

NO!

Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Observe that v_1, v_2 is a linearly independent set of vectors; v_1, v_3 is a linearly independent set of vectors; and v_2, v_3 is a linearly independent set of vectors; but v_1, v_2, v_3 is linearly dependent because $v_1 + v_2 = v_3$.