Math 544, Exam 1, Fall, 2020

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, send a picture of your solutions to

kustin@math.sc.edu

You should KEEP this piece of paper. If possible: put the problems in order before you take your picture. (Use as much paper as necessary).

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be an $n \times n$ matrix. Suppose the vectors v_1, v_2, v_3 are linearly independent. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

NO! Here is an example.

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is clear that v_1, v_2, v_3 are linearly independent. It is also clear that

$$Mv_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad Mv_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad Mv_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly dependent because

$$1Mv_1 + 0Mv_2 + 0Mv_3 = 0$$

is a non-trivial linear combination of Mv_1, Mv_2, Mv_3 which is equal to 0.

(2) **Let v_1, v_2, and v_3 be vectors in $\mathbb{R}^n$ and M be a nonsingular $n \times n$ matrix. Suppose the vectors v_1, v_2, v_3 are linearly independent. Do the vectors Mv_1, Mv_2, Mv_3 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

The vectors Mv_1, Mv_2, Mv_3 are linearly independent.

Proof. Suppose c_1, c_2, c_3 are numbers with

$$c_1 M v_1 + c_2 M v_2 + c_3 M v_3 = 0.$$

Use the property of scalars and the fact that matrix multiplication distributes over addition to see that

$$M(c_1 v_1 + c_2 v_2 + c_3 v_3) = 0.$$

The matrix M is nonsingular; hence, the only vector w with $Mw = 0$ is $w = 0$. Thus, $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. On the other hand, the vectors v_1, v_2, v_3 are linearly independent. It follows that c_1, c_2, c_3 must all be zero. We have proven that Mv_1, Mv_2, Mv_3 are linearly independent. $\square$

(3) **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

> The vectors $v_1, \dots v_p$ in $\mathbb{R}^n$ are <u>linearly independent</u> if the only numbers $c_1, \dots, c_p$ with $c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0$ are $c_1 = c_2 = \dots = c_p = 0$.

(4) **Suppose A and B are $n \times n$ matrices. Is $(A - B)(A + B) = A^2 - B^2$? If yes, prove it. If no, give a counterexample.**

The answer is no. If A and B don't commute then $(A - B)(A + B)$ is different than $A^2 - B^2$. We give an example of two 2×2 matrices that do not commute and we verify that $(A - B)(A + B) \neq A^2 - B^2$ for this pair of matrices. Well, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We compute

$$(A - B)(A + B) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

On the other hand,

$$A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

> We have exhibited matrices A and B with $(A - B)(A + B) \neq A^2 - B^2$.

(5) **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations. $\boxed{CIRCLE}$ your answer.**

$$A = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix}.$$

We apply row operations to

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 1 & 8 & 3 \\ 1 & 4 & 5 & 2 & 10 & 5 \\ 3 & 12 & 15 & 4 & 26 & 11 \end{array}\right].$$

Replace $R2 \mapsto R2 - R1$ and $R3 \mapsto R3 - 3R1$ to get

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 1 & 8 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right].$$

Replace $R1 \mapsto R1 - R2$ and $R3 \mapsto R3 - R2$ to get

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution of the system of equations is

$$\boxed{\begin{array}{c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \\ \text{where } x_2, x_3, \text{ and } x_5 \text{ are free to take any value.} \end{array}}$$

To obtain particular solutions of the system of equations, take $x_2 = x_3 = x_5 = 0$ to obtain v_1; $x_2 = 1$, $x_3 = x_5 = 0$ to obtain v_2; $x_2 = x_5 = 0$, $x_3 = 1$ to obtain v_3 and $x_2 = x_3 = 0$, $x_5 = 1$ to obtain v_4:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

We check

$$Av_1 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \ \checkmark$$

$$Av_2 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \ \checkmark$$

$$Av_3 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b, \ \checkmark$$

and

$$Av_4 = \begin{bmatrix} 1 & 4 & 5 & 1 & 8 \\ 1 & 4 & 5 & 2 & 10 \\ 3 & 12 & 15 & 4 & 26 \end{bmatrix} \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 11 \end{bmatrix} = b. \ \checkmark$$