Math 544, Exam 1, Spring 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. SHOW your work. **No Calculators or Cell phones.** Write your answers as legibly as you can. Make your work be coherent and clear. Write in complete sentences. I will post the solutions on my website shortly after the exam is finished.

1. **Find the GENERAL solution of the system of linear equations $Ax = b$. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations. $\boxed{CIRCLE}$ your answer.**

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 & 4 & 0 \\ 2 & 4 & 6 & 1 & 4 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We apply the row operations $R2 \mapsto R2 - R1$ and $R3 \mapsto R3 - 2R1$ to

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 0 & 0 & 0 & -1 \\ 1 & 2 & 3 & 1 & 4 & 0 & 0 \\ 2 & 4 & 6 & 1 & 4 & 1 & 1 \end{array}\right]$$

to obtain

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 1 & 3 \end{array}\right].$$

Apply the row operation $R3 \mapsto R3 - R2$ to obtain

$$\left[\begin{array}{cccccc|c} 1 & 2 & 3 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right].$$

The general solution of the system of equations is

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, x_2,\ x_3, \text{ and } x_5 \text{ are arbitrary} \right\}}$$

Some specific solutions are

$$\boxed{\left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -3 \\ 1 \\ 2 \end{bmatrix} \right\}}$$

(In v_1 we took $x_2 = x_3 = x_5 = 0$. In v_2 we took $x_2 = 1$ and $x_3 = x_5 = 0$. In v_3 we took $x_3 = 1$ and $x_2 = x_5 = 0$. In v_4 we took $x_5 = 1$ and $x_2 = x_3 = 0$.) Observe that $Av_i = b$ for $1 \le i \le 4$.

2. **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1, \dots, c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1 = c_2 = \cdots = c_p = 0$.

3. **Consider the system of equations $Ax = b$ where $A = \begin{bmatrix} 1 & -a \\ a & -1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $b = \begin{bmatrix} 3 \\ 4a-1 \end{bmatrix}$.**

We apply the row operation $R2 \mapsto R2 - aR1$ to

$$\left[\begin{array}{cc|c} 1 & -a & 3 \\ a & -1 & 4a-1 \end{array}\right]$$

to obtain

$$(*) \qquad \left[\begin{array}{cc|c} 1 & -a & 3 \\ 0 & a^2-1 & a-1 \end{array}\right]$$

We see that if $a^2 - 1$ is not zero, then the system of equations has a unique solution. In other words, if a is not equal to 1 or -1, then the system has a unique solution. If $a = 1$, then the matrix (*) is

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

and the system of equations has many solutions. If $a = -1$, then the matrix (*) is

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & -2 \end{array}\right]$$

and the system of equations does not have any solutions.

(a) **For which values of a does the system of equations have no solution?** If $a = -1$, then the system of equations does not have any solutions.

(b) **For which values of a does the system of equations have exactly one solution?** If a is not equal to 1 or -1, then the system has a unique solution.

(c) **For which values of a does the system of equations have more than one solution?** If $a = 1$, then the system of equations has many solutions.

4. **Let v_1, v_2, v_3, v_4 be linearly independent vectors in $\mathbb{R}^m$, for some m. Do the vectors v_1, v_2, v_3 have to be linearly independent? If yes, prove the statement. If no give an example.**

YES. Suppose a_1, a_2 and a_3 are numbers with $a_1v_1 + a_2v_2 + a_3v_3 = 0$. Let $a_4 = 0$. We also have $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$. The vectors v_1, v_2, v_3, v_4 are linearly independent; hence all four coefficients a_1, a_2, a_3, a_4 must be zero. In particular, the coefficients a_1, a_2, and a_3 must be zero and the vectors v_1, v_2, v_3 are linearly independent.

5. **Let v_1, v_2, v_3, v_4 be linearly dependent vectors in $\mathbb{R}^m$, for some m. Do the vectors v_1, v_2, v_3 have to be linearly dependent? If yes, prove the statement. If no give an example.**

NO. Consider the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We see that v_1, v_2, v_3, v_4 are linearly dependent

$$(\text{because} \quad 1v_1 + 1v_2 + 1v_3 - v_4 = 0)$$

but v_1, v_2, v_3 are linearly independent because if $c_1v_1 + c_2v_2 + c_3v_3 = 0$, then

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all three coefficients c_i must be zero.