PRINT Your Name:________________________

Math 544, Exam 1, Fall 2009, Solution

Write your answers as legibly as you can.

Make your work be coherent and clear. Write in complete sentences

There are 12 problems on 6 pages. Problems 1, 2, 3, 4 are worth 9 points each. Each of the other problems is worth problem is worth 8 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators. No phones.**

I will post the solutions on my website shortly after the exam is finished.

1. **Find the GENERAL solution of the following system of linear equations. Also, list three SPECIFIC solutions, if possible. CHECK that the specific solutions satisfy the equations.**

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & & & & & - & x_5 & = & 1 \\ & & x_2 & + & 2x_3 & + & x_4 & + & 3x_5 & = & 1 \\ x_1 & & & - & x_3 & + & x_4 & + & x_5 & = & 0 \end{array}$$

Start with the matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 1 & 0 & -1 & 1 & 1 & 0 \end{array}\right].$$

Apply $R_3 \mapsto R_3 - R_1$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & -1 & -1 & 1 & 2 & -1 \end{array}\right].$$

Apply $R_1 \mapsto R_1 - R_2$ and $R_3 \mapsto R_3 + R_2$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & -1 & -4 & 0 \\ 0 & 1 & 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

Apply $R_1 \mapsto R_1 + 2R_3$ and $R_2 \mapsto R_2 - 2R_3$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 6 & 0 \\ 0 & 1 & 0 & -3 & -7 & 1 \\ 0 & 0 & 1 & 2 & 5 & 0 \end{array}\right].$$

This matrix is in reduced row echelon from. The solution set is the set of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ such that

$$\begin{aligned} x_1 &= {-3x_4} - 6x_5 \\ x_2 &= 1+3x_4 + 7x_5 \\ x_3 &= {-2x_4} - 5x_5 \end{aligned}$$

such that x_4 and x_5 are arbitrary. A different way to say this is to say that the solution set is

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ 7 \\ -5 \\ 0 \\ 1 \end{bmatrix} \middle| \, x_4, x_5 \in \mathbb{R} \right\}}$$

Check. Our answer is correct. When $x_4 = x_5 = 0$ our answer is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and this proposed solution works because

$$\begin{aligned} 1 &= 1 \\ 1 &= 1 \\ 0 &= 0. \checkmark \end{aligned}$$

When $x_4 = 1$ and $x_5 = 0$ our answer is

$$\begin{bmatrix} -3 \\ 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

and this proposed solution works because

$$\begin{aligned} &-3+4=1\\ &4-4+1=1\\ &-3+2+1=0.\checkmark \end{aligned}$$

When $x_4=0$ and $x_5=1$ our answer is

$$\begin{bmatrix} -6\\ 8\\ -5\\ 0\\ 1 \end{bmatrix}$$

and this proposed solution works because

$$\begin{aligned} &-6+8-1=1\\ &8-10+3=1\\ &-6+5+1=0.\checkmark \end{aligned}$$

2. **Define "linearly independent". Use complete sentences. Include everything that is necessary, but nothing more.**

The vectors $v_1,\dots,v_p$ in $\mathbb{R}^n$ are *linearly independent* if the only numbers $c_1,\dots,c_p$ with $\sum_{i=1}^{p} c_i v_i = 0$ are $c_1=c_2=\cdots=c_p=0$.

3.

(a) **Define "non-singular". Use complete sentences. Include everything that is necessary, but nothing more.**

The square matrix A is non-singular if the **only** column vector x with $Ax=0$ is $x=0$.

(b) **Let A be an $n\times n$ matrix. List three statements that are equivalent to the statement "A is non-singular".**

1. The columns of A are linearly independent.
2. For every vector b in $\mathbb{R}^n$, the system of equations $Ax=b$ has a unique solution.
3. The matrix A is invertible.

4. **Let A and B be 2×2 matrices with A not equal to the zero matrix and $A^2 = AB$. Does A have to equal B? If yes, prove your answer. If no, give a counterexample.**

$\boxed{\text{NO.}}$ Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 75 & 82 \\ 0 & 0 \end{bmatrix}$. We see that A is not the zero matrix and A does not equal B; but, A^2 and AB are both equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

5. **Let A and B be 2×2 matrices. Does $(A+B)^2$ have to equal $A^2+2AB+B^2$? If yes, prove your answer. If no, give a counterexample.**

NO. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. We see that

$$(A+B)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

On the other hand,

$$\begin{aligned} A^2+2AB+B^2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}; \end{aligned}$$

hence, $(A+B)^2 \neq A^2+2AB+B^2$.

6. **Recall that the matrix A is called symmetric if $A^{\mathrm{T}} = A$. Let A and B be 2×2 symmetric matrices. Does AB have to be a symmetric matrix? If yes, prove your answer. If no, give a counterexample.**

NO. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We see that A and B are each symmetric, but the product

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not symmetric.

7. **Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^4$ with v_1, v_2, v_3 linearly dependent. Do v_1, v_2, v_3, v_4 have to be linearly dependent? If yes, prove your answer. If no, give a counterexample.**

YES. The first sentence guarantees that there are numbers a_1, a_2, a_3, at least one of which is non-zero, with $a_1v_1 + a_2v_2 + a_3v_3 = 0$. Thus, we have numbers $a_1, a_2, a_3, 0$, at least one of which is not zero, and $a_1v_1 + a_2v_2 + a_3v_3 + 0v_4 = 0$. We conclude that the vectors v_1, v_2, v_3, v_4 are linearly dependent.

8. **Let v_1, v_2, v_3, v_4 be vectors in $\mathbb{R}^4$ with v_1, v_2, v_3 linearly independent. Do v_1, v_2, v_3, v_4 have to be linearly independent? If yes, prove your answer. If no, give a counterexample.**

NO. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We have v_1, v_2, v_3 linearly indepndent, but v_1, v_2, v_3, v_4 linearly dependent (since $1v_1 + 1v + 2 + 0v_3 - v_4 = 0$.)

9. **Suppose v_1, v_2 and v_3 are vectors in $\mathbb{R}^4$ with v_1, v_2 linearly independent, v_1, v_3 linearly independent, and v_2, v_3 linearly independent. Do the vectors v_1, v_2, v_3 have to be linearly independent? If yes, give a proof. If no, give an example.**

NO. Consider the vectors:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The vectors v_1 and v_2 are independent (since neither vector is a multiple of the other). The vectors v_1 and v_3 are independent (since neither vector is a multiple of the other). The vectors v_2 and v_3 are independent (since neither vector is a multiple of the other). Yet the vectors v_1, v_2, v_3 are dependent because $-2v_1 + v_2 + v_3 = 0$.

10. **Suppose A is a 2×3 matrix and B is a 3×2 matrix with $AB = I$. Does BA have to equal I? If yes, give a proof. If no, give an example.**

NO. Take

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We see

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

but

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

11. **Suppose A and B are 2×2 matrices with $AB = I$. Does BA have to equal I? If yes, give a proof. If no, give an example.**

YES. The hypothesis $AB = I$ guarantees that B is non-singular because if x is a vector with $Bx = 0$, then we multiply both sides by A to learn $x = ABx = A0 = 0$. The only vector that B sends to zero is $x = 0$. The non-singular matrix theorem (see problem 3) guarantees that B is invertible. So there is a matrix B^{-1} with $BB^{-1} = I$ and $B^{-1}B = I$. Multiply both sides of $AB = I$ on the right with B^{-1} to get

$$A = ABB^{-1} = IB^{-1} = B^{-1}.$$

We conclude that

$$BA = BB^{-1} = I.$$

12. **Suppose v_1, v_2, v_3 are linearly independent vectors in $\mathbb{R}^4$. Do the vectors $v_1 + v_2$, $v_2 - v_3$, $v_3 + v_1$ have to be linearly independent? If yes, give a proof. If no, give an example.**

NO. The vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent, but

$$v_1 + v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 - v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_3 + v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly dependent because

$$1(v_1 + v_2) - 1(v_2 - v_3) - 1(v_3 + v_1) = 0$$

and the coefficients 1, -1, -1 are not all zero.