Math 544, Fall 2009, Exam 3, Solution

Use my paper. **Please turn the problems in order. Please leave 1 square inch in the upper left hand corner for the staple.**

The exam is worth 50 points. There are 8 problems. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

1. (6 points) **Define "linear transformation". Use complete sentences.**

The function T from the vector space V to the vector space W is a *linear transformation* if $T(v + v') = T(v) + T(v')$ for all v and v' in V and $T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{R}$.

2. (6 points) **The *trace* of the square matrix A is the sum of the numbers on its main diagonal. Let V be the set of all 3×3 matrices with trace 0. Is the set V a vector space? Explain.**

YES. We show that V is closed under addition and scalar multiplication. Take A and B in V and scalars a and b. The entry of A in row i and column j is denoted $A_{i,j}$. The fact that A and B are in V tells us that $\sum\limits_{i=1}^{3} A_{i,i} = \sum\limits_{i=1}^{3} B_{i,i} = 0$. We study

$$\sum_{i=1}^{3}(aA + bB)_{i,i} = \sum_{i=1}^{3}(aA_{i,i} + bB_{i,i}) = a\sum_{i=1}^{3} A_{i,i} + b\sum_{i=1}^{3} B_{i,i} = 0.$$

We conclude that $aA + bB$ is in V and therefore, V is a vector space.

3. (6 points) **Let $A = \begin{bmatrix} 7 & 6 \\ -3 & -2 \end{bmatrix}$. Find a matrix B with $B^2 = A$. Check your answer.**

One quickly computes that the eigenvalues of A are 1 and 4 and that the corresponding eigenvectors of A are $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Indeed, $Av_1 = v_1$ and $Av_2 = 4v_2$. So $AS = SD$, where

$$S = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

We take

$$B = S\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}S^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}}.$$

Check. We see that

$$B^2 = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ -3 & -2 \end{bmatrix}. \checkmark$$

4. (6 points) **Find an orthogonal set which is a basis for the null space of** $A = [1 \quad -1 \quad 1 \quad -1]$**. Check your answer.**

One basis for the null space of A is

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $u_1 = v_1$. Let

$$u_2' = v_2 - \frac{v_2^{\mathrm{T}} u_1}{u_1^{\mathrm{T}} u_1} u_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}.$$

Let

$$u_2 = 2u_2' = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

Be sure to check that u_2 is in the null space of A and u_2 is perpendicular to u_1. Let

$$u_3' = v_3 - \frac{v_3^{\mathrm{T}} u_1}{u_1^{\mathrm{T}} u_1} u_1 - \frac{v_3^{\mathrm{T}} u_2}{u_2^{\mathrm{T}} u_2} = \frac{1}{6} \begin{bmatrix} 2 \\ -2 \\ 2 \\ 6 \end{bmatrix}$$

Let

$$u_3 = 3u_3' = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

Be sure to check that u_3 is in the null space of A and u_3 is perpendicular to u_1 and u_2. Our orthogonal set is:

$$\boxed{u_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad u_2 = \begin{bmatrix}-1\\1\\2\\0\end{bmatrix} \quad \text{and} \quad u_3 = \begin{bmatrix}1\\-1\\1\\3\end{bmatrix}}.$$

5. **Let $T\colon V \to W$ be a linear transformation of vector spaces. Suppose that the vectors $v_1 \dots, v_a$ in V are a basis for the null space of T and that $w_1, \dots, w_b$ in W are a basis for the image of T. Let $u_1, \dots, u_b$ in V be vectors with $T(u_i) = w_i$ for $1 \le i \le b$.**
   (a) (4 points) **Prove that $v_1, \dots, v_a, u_1, \dots, u_b$ span V.**

Let v be an arbitrary vector of V. We know that $T(v)$ is in the image of T and $w_1, \dots, w_b$ span the image of T; so $T(v) = \sum\limits_{i=1}^{b} c_i w_i$ for some constants c_i. Of course, $w_i = T(u_i)$; so, $T(v) = \sum\limits_{i=1}^{b} c_i T(u_i)$ and $T(v - \sum\limits_{i=1}^{b} c_i u_i) = 0$ and $v - \sum\limits_{i=1}^{b} c_i u_i$ is in the null space of T. The null space of T is spanned by $v_1 \dots, v_a$; hence there are constants $d_1 \dots, d_a$ with $v - \sum\limits_{i=1}^{b} c_i u_i = \sum\limits_{i=1}^{a} d_i v_i$. Thus, v has been expressed as a linear combination of $v_1, \dots, v_a, u_1, \dots, u_b$. We conclude that $v_1, \dots, v_a, u_1, \dots, u_b$ span V.

   (b) (4 points) **Prove that $v_1, \dots, v_a, u_1, \dots, u_b$ are linearly independent vectors in V.**

Suppose that

$$(*) \qquad \sum_{i=1}^{b} c_i u_i + \sum_{i=1}^{a} d_i v_i = 0.$$

Apply the linear transformation T to see that

$$\sum_{i=1}^{b} c_i T(u_i) + \sum_{i=1}^{a} d_i T(v_i) = 0.$$

We know that $T(u_i) = w_i$ and $T(v_i) = 0$. Thus, $\sum\limits_{i=1}^{b} c_i w_i = 0$. But $w_1, \dots, w_b$ are linearly independent; hence, $c_1 = \dots = c_b = 0$. Now, (*) says that $\sum\limits_{i=1}^{a} d_i v_i = 0$. But $v_1, \dots, v_a$ are linearly independent and $d_1 = \dots = d_a = 0$.

6. (6 points) **Let**

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \quad \textbf{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve $Ax = b$. (You may do the problem any way you wish; however, you may find it helpful to notice that the columns of A form an orthogonal set.) Check your answer.

Multiply both sides of $Ax = b$ by A^{T} to see that

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x = \begin{bmatrix} 10 \\ -1 \\ -1 \\ 4 \end{bmatrix}.$$

Conclude that

$$\boxed{x = \begin{bmatrix} 5/2 \\ -1/2 \\ -1/2 \\ 1 \end{bmatrix}}.$$

Check. $Ax = b$. ✓

7. (6 points) **Let $W = \{f \colon \mathbb{R} \to \mathbb{R} \mid f$ is differentiable$\}$. Is W a vector space? Explain.**

Yes. The set W is closed under addtion and scalar multiplication. The sum of two differentiable functions is differentiable. A constant multiplre of a diffetrentiable function is differentiable.

8. (6 points) **Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = \frac{1}{\sqrt{3}}x$. Find a matrix M with $T(v) = Mv$ for all vectors $v \in \mathbb{R}^2$.**

The line ℓ, which is $y = \frac{1}{\sqrt{3}}x$, makes the angle $\theta = \pi/6$ with the x-axis. The matrix for T is

$$M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -1/2 \end{bmatrix}}.$$

Check. Take a vector on the line ℓ, like $v = \begin{bmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$. Observe that $Mv = v$.

Take a vector perpendicular to ℓ, like $v = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -1 \end{bmatrix}$. Observe that $Mv = -v$.