Fall 242 Final 1994

PRINT Your Name:____________________________

There are 15 problems on 9 pages. Problem 1 is worth 16 points each. Each of the other problems is worth 6 points. The exam is worth a total of 100 points. SHOW your work.

CIRCLE your answer. CHECK your answer, whenever possible.

1.

(a) State the Existence and Uniqueness Theorem for first order differential equations.

Consider the Initial Value Problem $y' = f(x,y)$ $y(a) = b$.

a) If there is a rectangle R in the (x,y)-plane which contains (a,b) and on which f(x,y) is continuous, then the IVP has at least one solution which is defined on some interval which contains $x=a$

b) If f_y is also continuous on R, then the IVP has a unique solution which is defined on some (possibly smaller) interval which contains $x=a$.

(b) What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem

$$(1+x^2)y' = (y-3)^2 \quad y(0) = 3?$$

$y' = \frac{(y-3)^2}{1+x^2}$ so $f(x,y) = \frac{(y-3)^2}{1+x^2}$ $f_y = \frac{2(y-3)}{1+x^2}$

f and f_y are continuous everywhere.

So the IVP has a unique solution

(c) Solve the Initial Value Problem of part (b).

$y = 3$

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2. A 400 gallon tank initially contains 200 gallons of brine containing 30 pounds of salt. Brine containing 3 pounds of salt per gallon enters the tank at the rate of 6 gal./sec., and the mixed brine in the tank flows out at the rate of 2 gal./sec.. How much salt will the tank contain at the moment it becomes full?

$X(t)$ = # of lbs of salt in the tank at time t

$X(0) = 30$

$\frac{dx}{dt}$ = Rate In − Rate out $= 3\frac{\text{lbs}}{\text{gal}}\, 6\,\text{gal}/\text{sec} - \frac{X \text{ lbs}}{200+4t \text{ gal}}\, 2\,\frac{\text{gal}}{\text{sec}}$.

$$\frac{dx}{dt} = 18 - \frac{2x}{200+4t}$$

$$\frac{dx}{dt} + \frac{2}{200+4t}x = 18$$

$$\mu = e^{\int \frac{2}{200+4t}dt} = e^{\frac{1}{2}\ln(200+4t)} = \sqrt{200+4t}$$

$$\frac{dx}{dt}\sqrt{200+4t} + \frac{2}{\sqrt{200+4t}}x = 18\sqrt{200+4t}$$

$$x\sqrt{200+4t} = 18\int\sqrt{200+4t}\,dt$$

$$x\sqrt{200+4t} = \frac{18}{4}\left(\frac{2}{3}\right)(200+4t)^{\frac{3}{2}} + c$$

$$X(t) = 3(200+4t) + \frac{c}{\sqrt{200+4t}}$$

$$30 = 600 + \frac{c}{\sqrt{200}}$$

$$-570\sqrt{200} = c$$

$$X(t) = 3(200+4t) - \frac{570\sqrt{200}}{\sqrt{200+4t}}$$

$$X(50) = 3(400) - \frac{570\sqrt{200}}{20}$$

$$= \boxed{796.95 \text{ lbs}}$$

3. Find all solutions of $x^2y' = xy + x^2e^{y/x}$.

$y' = \frac{y}{x} + e^{\frac{y}{x}}$ $\quad V = \frac{y}{x}$

$xv = y$

$xv' + v = y'$

$$xv' + \cancel{v} = \cancel{v} + e^v$$

$$e^{-v}\,dv = \frac{dx}{x}$$

$$-e^{-v} = \ln x + c$$

$$e^{-v} = -\ln x + k$$

$$-v = \ln(k - \ln x)$$

$$\boxed{y = -x\ln(k - \ln x)}$$

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4. Find all solutions of $xy' + (2x-3)y = 4x^4$.

$y' + \left(2 - \frac{3}{x}\right) y = 4x^3$

$\mu = e^{\int 2 - \frac{3}{x}\,dx} = e^{2x - 3\ln x} = \frac{e^{2x}}{x^3}$

$\frac{e^{2x}}{x^3} y' + \left(2 - \frac{3}{\cancel{x}}\right) \frac{e^{2x}}{x^3} = 4e^{2x}$

$\frac{e^{2x}}{x^3} y = 2e^{2x} + c$

$y = 2x^3 + cx^3 e^{-2x}$

5. Find all solutions of $y'' + 2y' + y = 0$.

$r^2 + 2r + 1 = 0$

$(r+1)^2 \Rightarrow r = -1, -1$

$y = c_1 e^{-x} + c_2 x e^{-x}$

6. Find all solutions of $y'' - 4y' + 13y = 0$.

$r^2 - 4r + 13 = 0$

$r = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$

$y = e^{2x}\left[c_1 \sin 3x + c_2 \cos 3x\right]$

7. Find all solutions of $y'' - 3y' + 2y = 10 \sin x$.

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 2, 1$

Solution to homog problem is $y = c_1 e^{2x} + c_2 e^{x}$

$y = A \sin x + B \cos x$

$y' = A \cos x - B \sin x$

$y'' = -A \sin x - B \cos x$

$-A \sin x - B \cos x$

$+3B \sin x \; 3A \cos x$

$+2A \sin x + 2B \cos x = 10 \sin x$

$A + 3B = 10$

$-3A + B = 0$

$10A = 10$

$A = 1$

$B = 3$

$y = c_1 e^{2x} + c_2 e^{x} + \sin x + 3 \cos x$

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8. Find all solutions of $y'' - 3y' + 2y = e^x$.

homog prob: $r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

Solution of homog. problem is $y = c_1 e^x + c_2 e^{2x}$

Try $y = Axe^x$ $\quad y' = A(xe^x + e^x)$ $\quad y'' = A(xe^x + 2e^x)$

$$A\begin{pmatrix} xe^x + 2e^x \\ -3xe^x - 3e^x \\ 2xe^x \end{pmatrix} = e^x \quad \therefore A = -1$$

$$y_1 = c_1 e^x + c_2 e^{2x} - xe^x$$

9. Find all solutions of $x^2y'' - 4xy' + 6y = 0$.

Try $y = x^r$ $\quad y' = rx^{r-1}$ $\quad y'' = r(r-1)x^{r-2}$

$x^r(r(r-1) - 4r + 6) = 0$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r = 2, 3$

$$y = c_1 x^2 + c_2 x^3$$

10. Find all solutions of $y' = \sin 2x \cos 3x$.

$$y = \int \sin 2x \cos 3x \, dx$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] = \sin x \cos y$$

$$y = \frac{1}{2}\int \sin 5x + \sin(-x)\, dx = \frac{1}{2}\left[-\frac{\cos 5x}{5} + \cos x\right] + C = y$$

11. Find all solutions of $y'' + y = \tan x$.

The solutions of the homog equation are $y = c_1 \cos x + c_2 \sin x$.
One solution of the non-homogeneous equation is $y = u_1 \cos x + u_2 \sin x$
where $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix}$ so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}\begin{bmatrix} 0 \\ \tan x \end{bmatrix}$

$$u_1' = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

$$u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2' = \sin x \quad u_2 = -\cos x$$

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$$

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12. Find $\mathcal{L}^{-1}\left(\ln \frac{s-3}{s+2}\right)$. Let $f(t) = \mathcal{L}^{-1}\left(\ln \frac{s-3}{s+2}\right)$

so $\mathcal{L}(f(t)) = \ln(s-3) - \ln(s+2)$

so $-\mathcal{L}(t\,f(t)) = \frac{d}{ds}\mathcal{L}(f(t)) = \frac{1}{s-3} - \frac{1}{s+2} = \mathcal{L}\left(e^{3t} - e^{-2t}\right)$

$$f(t) = \frac{-\left(e^{3t} - e^{-2t}\right)}{t}$$

13. Find $\mathcal{L}^{-1}\left(\frac{5-2s}{s^2+7s+10}\right)$. $=$

$$\mathcal{L}^{-1}\left(\frac{3}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{-5}{s+5}\right)$$

$$= 3e^{-2t} - 5e^{-5t}$$

14. Find one nontrivial solution of

$$\begin{cases} tx'' - 2x' + tx = 0, \\ x(0) = 0. \end{cases}$$

$\mathcal{L}(x) = X$ $\qquad \mathcal{L}(tx) = -X'$

$\mathcal{L}(x') = sX$ $\qquad \mathcal{L}(tx'') = -2sX - s^2X'$

$\mathcal{L}(x'') = s^2X - x''(0)$

$-2sX - s^2X' - 2sX - X' = 0$

$-(s^2+1)X' = 4sX$

$\frac{dX}{X} = \frac{-4s}{s^2+1}\,ds$

$\ln X = -2\ln(s^2+1)$

$X = \frac{1}{(s^2+1)^2}$

$x = \frac{1}{2}(\sin t - t\cos t)$

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15. Solve

$$\begin{cases} x'' + 4x = \begin{cases} t & \text{if } 0 \le t < 1, \\ 0 & \text{if } 1 \le t, \end{cases} \\ x(0) = 0, \quad \text{and} \quad x'(0) = 0. \end{cases}$$

Let $X = \mathcal{L}(x)$
so $\mathcal{L}(x') = sX$
$\mathcal{L}(x'') = s^2 X$

Let $f(t) = \begin{cases} t & \text{for } 0 \le t < 1 \\ 0 & \text{for } t \le 1 \end{cases}$

so $f(t) = [1 - u(t-1)](t) = t - u(t-1)(t-1) - u(t-1)$

$$\mathcal{L}(f(t)) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} + e^{-s}\frac{1}{s}$$

$$X = \frac{1}{s^2(s^2+4)} - e^{-s}\left[\frac{1}{s^2(s^2+4)}\right] - e^{-s}\frac{1}{s(s^2+4)}$$

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right] \qquad \frac{1}{s(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$x(t) = \frac{1}{4}\left[t - \frac{1}{2}\sin 2t - u(t-1)\left[t-1-\frac{1}{2}\sin 2(t-1)\right] - u(t-1)\left[1 - \cos 2(t-1)\right]\right]$$

$$x(t) = \frac{1}{4}\begin{cases} t - \frac{1}{2}\sin 2t & \text{if } 0 \le t < 1 \\ t - \frac{1}{2}\sin 2t - \left(t-1-\frac{1}{2}\sin 2(t-1)\right) + 1 - \cos 2(t-1) & \text{if } 1 \le t \end{cases}$$

$$x(t) = \frac{1}{4}\begin{cases} t - \frac{1}{2}\sin 2t & \text{if } 0 \le t < 1 \\ -\frac{1}{2}\sin 2t + \frac{1}{2}\sin(2t-2) + \cos(2t-2) & \text{if } 1 \le t \end{cases}$$

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