1993 Final Exam Math 242

① Consider the initial value problem $y''+P(x)y'+Q(x)y=f(x)$, $y(a)=b_0$ and $y'(a)=b_1$. If P, Q and f are continuous on some interval I which contains $x=a$, then there is a unique solution y to the IVP which is defined on all of I.

② Try solutions of the form $y=e^{rx}$ for the homogeneous problem: $r^2-5r+6=0$ $(r-2)(r-3)=0$. For the non-homogeneous problem try $y=Ae^x$:

$A(e^x-5e^x+6e^x)=e^x$ so $A=\frac{1}{2}$

$$y=c_1e^{2x}+c_2e^{3x}+\frac{1}{2}e^x$$

③ Try solutions of the form $y=x^r$

$$x^2(r-1)rx^{r-2}-3xrx^{r-1}+3x^r=0$$

$$r^2-4r+3=0 \qquad (r-1)(r-3)=0$$

$$\boxed{y=c_1x+c_2x^3}$$

④ $y'-\frac{3}{x}y=x^3\cos x$

$\mu=e^{\int -\frac{3}{x}dx}=e^{-3\ln x}=\frac{1}{x^3}$

$\frac{1}{x^3}y'-\frac{3}{x^4}y=\cos x$

$\frac{d}{dx}\left(\frac{1}{x^3}y\right)=\cos x$

$\frac{1}{x^3}y=\sin x+C$

$y=x^3\sin x+x^3C$

$1=y(2\pi)=(2\pi)^3C$

$\frac{1}{8\pi^3}=c$

$$y=x^3\sin x+\frac{x^3}{8\pi^3}$$

⑤ $y'=\frac{-y(3x+y)}{x(x+y)}$ Let $v=\frac{y}{x}$ $xv=y$ $xv'+v=y'$

$xv'+v=-v\,\frac{3+v}{1+v}$ 6

$xv'=\frac{-3v-v^2}{1+v}-v$

$xv'=\frac{-3v-v^2-v-v^2}{1+v}=\frac{-4v-2v^2}{1+v}$

$\frac{1+v}{-2(2v+v^2)}\,dv=\frac{dx}{x}$

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$\therefore -\frac{1}{4}\ln(2v+v^2) = \ln x + c$

$\ln(2v+v^2) = -4\ln x + k$

$2v + v^2 = \ell\,\frac{1}{x^4}$

$2\frac{y}{x} + \frac{y^2}{x^2} = \frac{\ell}{x^4}$

$\boxed{2x^3y + x^2y^2 = \ell}$

(6) Let $X(t)$ = # of lbs of salt in the tank at time t.

$\frac{dX}{dt}$ = rate in − rate out $= 8\,\frac{\text{lb}}{\text{min}} - \frac{X}{90+t}\,\frac{\text{lb}}{\text{gal}}\cdot 3\,\frac{\text{gal}}{\text{min}}$

$\frac{dX}{dt} + \frac{3X}{90+t} = 8 \qquad X(0) = 90 \qquad$ Find $X(30)$.

$\mu = e^{\int \frac{3}{90+t}\,dt} = e^{+3\ln(90+t)} = (90+t)^3$

$(90+t)^3\frac{dx}{dt} + 3(90+t)^2 X = 8(90+t)^3$

$(90+t)^3 X = 2(90+t)^4 + C$

$X = 2(90+t) + \frac{C}{(90+t)^3}$

$90 = 2(90) + \frac{C}{(90)^3}$

$-(90)^4 = C$

$X(t) = 2(90+t) - \frac{(90)^4}{(90+t)^3}$

$X(30) = 2(120) - \frac{(90)^4}{(120)^3} = \boxed{202.03 \text{ pounds}}$

⑦ The solution of $y'' + 9y = 0$ is $c_1 \cos 3x + c_2 \sin 3x$

Try variation of parameters:

$y = u_1 \cos 3x + u_2 \sin 3x$ where

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$$\begin{bmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sec 3x \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3\cos 3x & -\sin 3x \\ 3\sin 3x & \cos 3x \end{bmatrix} \begin{bmatrix} 0 \\ 2\sec 3x \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -2\tan 3x \\ 2 \end{bmatrix} \qquad u_1 = \frac{2}{9} \ln|\cos 3x| \qquad u_2 = \frac{2}{3} x$$

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$$y = c_1 \cos 3x + c_2 \sin 3x + \frac{2}{9} \ln|\cos 3x| \cos 3x + \frac{2}{3} x \sin 3x$$

⑧ $\mathcal{L}(x) = X$

$\mathcal{L}(x') = sX - x(0) = sX$

$\mathcal{L}(x'') = s^2 X - x'(0)$

$\mathcal{L}(t x'') = -\frac{d}{ds} \mathcal{L}(x'') = -2sX - s^2 X'$

$\mathcal{L}(t x') = -\frac{d}{ds} \mathcal{L}(x') = -sX' - X$

$-2sX - s^2 X' - 3sX' \cancel{-3X} + \cancel{3X} - sX = 0$

$-(s^2 + 3s) \frac{dX}{ds} = 3sX$

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$$-\frac{dX}{X} = \frac{3s}{s(s+3)} \, ds$$

$$-\ln X = 3\ln(s+3) \qquad \text{(constant not needed)}$$

$$X = \frac{1}{(s+3)^3}$$ (4)

$$x = \mathcal{L}^{-1}(X) = t^2 e^{-3t}$$

$$x = t^2 e^{-3t}$$

(9) $x'' + 5x' + 4x = 1 - u(t-2)$

$$s^2 X + 5sX + 4X = \frac{1}{s}(1 - e^{-2s})$$

$$X = \frac{1}{s(s+4)(s+1)}(1-e^{-2s}) \overset{6}{=} \frac{1-e^{-2s}}{12}\left[\frac{3}{s} + \frac{-4}{s+1} + \frac{1}{s+4}\right]^{8}$$

$$x = \frac{1}{12}\left[3 - 4e^{-t} + e^{-4t} - u(t-2)\left[3 - 4e^{-(t-2)} + e^{-4(t-2)}\right]\right]$$

(10) $y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=0}^{\infty} n c_n x^{n-1} \qquad xy' = \sum_{n=0}^{\infty} n c_n x^n$

$$y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+2} + nc_n + c_n\right) x^n = 0$$

$\therefore (n+2)(n+1)c_{n+2} + (n+1)c_n = 0$ for all n

c_0 + c_1 are arbitrary $\quad c_{n+2} = \dfrac{-c_n}{n+2}$ 10 (31)

$c_2 = -\frac{c_0}{2}$	$c_4 = \frac{c_0}{2\cdot 4}$	$c_6 = -\frac{c_0}{2\cdot 4\cdot 6}$	etc.
$c_3 = -\frac{c_1}{3}$	$c_5 = \frac{c_1}{3\cdot 5}$	$c_7 = -\frac{c_1}{\ldots}$	

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$$y = c_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 4} - \frac{x^6}{\underset{48}{2\cdot 4\cdot 6}} + \frac{x^8}{\underset{384}{2\cdot 4\cdot 6\cdot 8}} - \cdots\right]$$

$$+ c_1\left[x - \frac{x^3}{3} + \frac{x^5}{3\cdot 5} - \frac{x^7}{\underset{105}{3\cdot 5\cdot 7}} + \frac{x^9}{\underset{945}{3\cdot 5\cdot 7\cdot 9}} - \cdots\right]$$