Math 242 1990 Final Exam

1) First order linear $\mu = e^{\int P(x)\,dx} = e^x$

$$e^x y' + e^x y = e^{2x}$$

$$\frac{d(e^x y)}{dx} = e^{2x}$$

$$e^x y = \frac{1}{2}e^{2x} + C$$

$$\boxed{y = \frac{1}{2}e^x + Ce^{-x}}$$

check: $y' + y = \frac{1}{2}e^x - Ce^{-x} + \frac{1}{2}e^x + Ce^{-x} = e^x$ ✓

2) Second order linear nonhomogeneous:

The solution of $y'' + y = 0$ is

$y = c_1 \sin x + c_2 \cos x$

Try the method of undetermined coefficients.

$y = Ae^{5x}$ $\quad y' = 5Ae^{5x}$ $\quad y'' = 25Ae^{5x}$

$$25A e^{5x} + Ae^{5x} = e^{5x}$$

$26A = 1 \qquad A = \frac{1}{26}$

$$\boxed{y = c_1 \sin x + c_2 \cos x + \frac{1}{26}e^{5x}}$$

3) $y = \sum_{n=0}^{\infty} c_n x^n \qquad y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$

$y'' + 4y = 0$ becomes $\sum_{n=0}^{\infty}\left((n+2)(n+1)c_{n+2} + 4c_n\right)x^n = 0$

$\therefore \quad c_{n+2} = \frac{-4}{(n+2)(n+1)}c_n$

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so c_0 and c_1 are arbitrary

$c_2 = \frac{-4}{2\cdot 1} c_0 \qquad c_3 = \frac{-4}{3\cdot 2} c_1 \qquad c_4 = \frac{-4}{4\cdot 3} c_2 = \frac{+(4)^2}{4!} c_0$

$c_5 = \frac{-4}{5\cdot 4} c_3 = \frac{+(4)^2}{5!} c_1 \qquad c_6 = \frac{-4}{6\cdot 5} c_4 = -\frac{(4)^3}{4!} c_0 \quad \cdots$

so $y = c_0\left(1 - \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4 - \frac{4^3}{6!}x^6 + \cdots\right)$

$+ c_1\left(x - \frac{4}{3!}x^3 + \frac{4^2}{5!}x^5 - \frac{4^3}{7!}x^7 + \cdots\right)$

$= c_0\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}x^r - \frac{(2x)^6}{6!}x^6 + \cdots\right)$

$+ \frac{c_1}{2}\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!}x^7 + \cdots\right)$

$$\boxed{y = c_0 \cos 2x + \frac{c_1}{2}\sin 2x}$$

(4) $4x^2y'' - 4xy' + 3y = 0$ is a Cauchy-Euler Equation

Try $y = x^r$. Get $x^r(4r(r-1) - 4r + 3) = 0$

$4r^2 - 8r + 3 = 0$

$(2r-1)(2r-3) = 0$

$r = \frac{1}{2}$ or $\frac{3}{2}$

Use Variation of Parameters to solve

$4x^2y'' - 4xy' + 3y = 8x^{\frac{4}{3}}$

$\rightarrow \quad y'' - \frac{1}{x}y' + \frac{3}{4x^2} = 2x^{-\frac{2}{3}} \quad \leftarrow$

$y_{partic} = u_1 x^{\frac{1}{2}} + u_2 x^{\frac{3}{2}}$ is a particular solution

where

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$$\begin{bmatrix} x^{\frac12} & x^{\frac32} \\ \frac12 x^{-\frac12} & \frac32 x^{\frac12} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2x^{-\frac23} \end{bmatrix}$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{x} \begin{bmatrix} \frac32 x^{\frac12} & -x^{\frac32} \\ -\frac12 x^{-\frac12} & x^{\frac12} \end{bmatrix} \begin{bmatrix} 0 \\ 2x^{-\frac23} \end{bmatrix} = \begin{bmatrix} -2x^{-\frac16} \\ 2x^{-\frac76} \end{bmatrix}$$

$$u_1 = -\frac{12}{5} x^{\frac56} \qquad u_2 = -12\, x^{-\frac16}$$

$$y_{partic} = -\frac{12}{5} x^{\frac56} \cdot x^{\frac12} - 12\, x^{-\frac16} x^{\frac32} = -\frac{72}{5} x^{\frac43}$$

check

$$4x^2 \left(-\frac{72}{5}\right)\left(\frac43\right) \frac13 x^{-\frac23} - 4x\left(-\frac{72}{5}\right)\frac43 x^{\frac13} + \left(-\frac{72}{5}\right) 3\, x^{\frac43}$$

$$= -\frac{72}{5} x^{\frac43} \left(\frac{16}{9} - \frac{48}{9} + \frac{27}{9} \right) = -\frac{72}{5} x^{\frac43} \left(\frac{-5}{9} \right) = 8 x^{\frac43}$$

$$\boxed{y = c_1 x^{\frac12} + c_2 x^{\frac32} - \frac{72}{5} x^{\frac43}}$$

5 Let $X = \mathcal{L}(x)$. $\qquad \mathcal{L}(t x') = -\frac{d}{ds}\mathcal{L}(x')$

$\mathcal{L}(x') = sX$

$\mathcal{L}(x'') = s^2 X - x'(0) \qquad = -(sX' + X)$

$$\mathcal{L}(t x'') = -\frac{d}{ds} \mathcal{L}(x'') = -(s^2 X' + 2sX)$$

$$-(s^2 X') - 2sX - sX' - \cancel{X} - 2sX + \cancel{X} = 0 \qquad \text{25}$$

$$(s^2 + s)X' + (2s + 2s)X = 0$$

$$X' = \frac{-4}{s+1} X$$

$$\frac{dX}{X} = \frac{-4}{s+1} ds$$

$$\ln X = -4 \ln s+1 + C$$

$$X = \frac{C_0}{(s+1)^4}$$

$$x = \mathcal{L}^{-1}(X) = \boxed{\frac{c_0}{6} e^{-t} t^3}$$

6 To solve

$X'' + 4X = 1 \quad X(0) = x'(0) = 0 \qquad 0 \le t \le \pi$:

$X = c_1 \sin 2t + c_2 \cos 2t + \frac{1}{4}$

where $x' = 2c_1 \cos 2t - 2c_2 \sin 2t$

$0 = c_2 + \frac{1}{4} \qquad c_2 = -\frac{1}{4}$

$0 = 2c_1 \qquad c_1 = 0$

$X = \frac{1}{4}(1 - \cos 2t) \qquad$ if $0 \le t \le \pi$

$\therefore X = \frac{1}{2} \sin^2 t \qquad$ if $0 \le t \le \pi$

$X(\pi) = 0$

$X'(\pi) = \sin t \cos t \big|_{\pi} = 0$

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To solve $X'' + 4X = 0 \quad X(\pi) = X'(\pi) = 0 \qquad \pi \le t$

$X(t) = 0 \quad$ for $\quad \pi \le t$

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$$\boxed{x = \begin{cases} \frac{1}{2}\sin^2 t & 0 \le t \le \pi \\ 0 & \pi \le t \end{cases}}$$