Math 242 1990 Exam 3

1

1) $x'' - 4x = 3t \quad x(0) = x'(0) = 0$

Let $X = \mathcal{L}(x)$.

Then $\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = sX$

$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s^2 X$

we know $\mathcal{L}(t) = \frac{1}{s^2}$

So $s^2 X - 4X = \frac{3}{s^2}$

$$X = \frac{3}{s^2(s^2-4)} = \frac{3}{4}\left(\frac{-1}{s^2} + \frac{1}{s^2-4}\right)$$

$$\therefore x = \mathcal{L}^{-1}(X) = \frac{-3}{4}t + \frac{3}{8}\sinh(2t)$$

$$x = \frac{3}{8}\left[\sinh(2t) - 2t\right]$$

2) $\mathcal{L}(f(t)) = \ln\left(\frac{s^2+2s+5}{(s+1)^2}\right)$ Find f.

$$\mathcal{L}(tf) = -\frac{d}{ds}\mathcal{L}(f) = -\frac{d}{ds}\left(\ln(s^2+2s+5) - 2\ln(s+1)\right)$$

$$= -\left[\frac{2s+2}{s^2+2s+5} - 2\frac{1}{s+1}\right]$$

$$= -\left[\frac{2(s+1)}{(s+1)^2+4} - 2\frac{1}{s+1}\right] = \mathcal{L}\left(-2e^{-t}\cos(2t) + 2e^{-t}\right)$$

$$f = \frac{1}{t}\left(-2e^{-t}\cos(2t) + 2e^{-t}\right)$$

11

(3) $tx'' - 2x' + tx = 0 \quad X(0) = 0$

Let $\mathbb{X} = \mathcal{L}(x)$

Then $\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = s\mathbb{X}$

$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s^2\mathbb{X} - x'(0)$

$$\mathcal{L}(tx'') = -\frac{d}{ds}\mathcal{L}(x'') = -\left(2s\mathbb{X} + s^2\mathbb{X}'\right)$$

$$\mathcal{L}(tx) = -\frac{d}{ds}\mathcal{L}(x) = -\mathbb{X}'$$

$$+2s\mathbb{X} + s^2\mathbb{X}' + 2s\mathbb{X} + \mathbb{X}' = 0$$

$$(s^2+1)\mathbb{X}' = -4s\mathbb{X}$$

$$\frac{d\mathbb{X}}{\mathbb{X}} = \frac{-4s}{s^2+1}\,ds$$

$$\ln\mathbb{X} = -2\ln(s^2+1) + C_0$$

$$\mathbb{X} = C_1\frac{1}{(s^2+1)^2}$$

$$x = \frac{C_1}{2}(\sin t - t\cos t)$$

$$\boxed{x = C(\sin t - t\cos t)}$$

4 $\quad 4x^2y'' - 4xy' + 3y = 8x^{\frac{4}{3}}.$

The corresponding homogeneous problem is an Euler-Cauchy equation; so we look for solutions of

$$(*) \quad 4x^2y'' - 4xy' + 3y = 0$$

of the form $y = x^r$

$$x^r\left(4r(r-1) - 4r + 3\right) = 0$$

$$4r^2 - 8r + 3 = 0$$

$$(2r-1)(2r-3) = 0 \qquad r = \tfrac{1}{2}, \tfrac{3}{2}$$

The gen. solution of $(*)$ is

$$y = c_1 x^{\frac{1}{2}} + c_2 x^{\frac{3}{2}}.$$

The method of variation of parameters tells us that

$$y_{partic} = u_1 x^{\frac{1}{2}} + u_2 x^{\frac{3}{2}}$$

is a particular solution of the original DE when

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{8x^{\frac{4}{3}}}{4x^2} \end{bmatrix}$$

!!

$$\begin{bmatrix} x^{\frac{1}{2}} & x^{\frac{3}{2}} \\ \frac{1}{2}x^{-\frac{1}{2}} & \frac{3}{2}x^{\frac{1}{2}} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2x^{-\frac{2}{3}} \end{bmatrix}$$

13

Multiply both sides by

$$\frac{1}{x}\begin{bmatrix} \frac{3}{2}x^{\frac{1}{2}} & -x^{\frac{3}{2}} \\ -\frac{1}{2}x^{-\frac{1}{2}} & x^{\frac{1}{2}} \end{bmatrix}$$ to get

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{x}\begin{bmatrix} \frac{3}{2}x^{\frac{3}{6}} & -x^{\frac{9}{6}} \\ -\frac{1}{2}x^{-\frac{3}{6}} & x^{\frac{3}{6}} \end{bmatrix}\begin{bmatrix} 0 \\ 2x^{-\frac{4}{6}} \end{bmatrix} = \frac{1}{x}\begin{bmatrix} -2x^{\frac{5}{6}} \\ 2x^{-\frac{1}{6}} \end{bmatrix}$$

$u_1' = -2x^{-\frac{1}{6}}$ $\quad u_1 = -2\left(\frac{6}{5}\right)x^{\frac{5}{6}}$

$u_2' = 2x^{-\frac{7}{6}}$ $\quad u_2 = 2(-6)\,x^{-\frac{1}{6}}$

$$Y_{partic} = -\frac{12}{5}\left(x^{\frac{5}{6}}\cdot x^{\frac{3}{6}} + 5x^{-\frac{1}{6}}x^{\frac{9}{6}}\right) = -\frac{72}{5}x^{\frac{4}{3}}$$

The Gen solution of the orig. DE is

$$\boxed{Y = c_1 x^{\frac{1}{2}} + c_2 x^{\frac{3}{2}} - \frac{72}{5}x^{\frac{4}{3}}}$$