1990 Exam 2 Math 242

1)

$yy'' = (y')^2$

Let $P(y) = y'$

$y'' = \frac{dy'}{dx} = \frac{dP}{dy}P$

$yP\frac{dP}{dy} = P^2$

$\frac{dP}{P} = \frac{dy}{y}$

$\ln|P| = \ln|y| + c$

$P = Ay$

$\frac{dy}{dx} = Ay$

$\frac{dy}{y} = A\,dx$

$\ln|y| = Ax + c_1$

$\boxed{y = Be^{Ax}}$

(2)

$x^2y'' - xy' + y = 0$

Try $y = x^r$

$x^r(r(r-1) - r + 1) = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

So $y = x$ is a solution

To find the other solution use reduction of order

$y = vx$

$y' = v + v'x$

$y'' = 2v' + v''x$

$2v'x^2 + v''x^3 - \cancel{xv} - v'x^2 + \cancel{vx} = 0$

$x^3v'' + x^2v' = 0$

integrating factor!

$xv'' + v' = 0$

$xv' = c$

$v' = \frac{c}{x}$

So $v = c\ln x$

general solution is $\boxed{y = c_1x + c_2x\ln x}$

(3) First solve $y'' + 2y' + y = 0$

try $y = e^{rx}$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1$

so $y = e^{-x}$ and $y = xe^{-x}$ are sols of the homog eq.

We try $y = Ax^2e^{-x} + Bx^3e^{-x}$ as our soln to the original equation

$y' = e^{-x}(2Ax + (3B-A)x^2 - Bx^3)$

$y'' = e^{-x}(2A + (6B-4A)x + (A-6B)x^2 + Bx^3)$

When we plug $y = Ax^2e^{-x} + Bx^3e^{-x}$ into the orig. DE we get

$$e^{-x}\left(\begin{array}{llll} 2A & +(6B-4A)x & +(A-6B)x^2 & +Bx^3 \\ & +4A\ x & (-2A+6B)x^2 & -2Bx^3 \\ & & +A\quad x^2 & +Bx^3 \end{array}\right) = xe^{-x}$$

$\therefore$ $2A = 0$

$6B = 1$

$\therefore$ $A = 0$ $B = \frac{1}{6}$

$$\boxed{y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{6}x^3e^{-x}}$$

4 $x'' + 2x' + 5x = 0$

try $x = e^{rt}$

$r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$x = e^{-t}(c_1\cos 2t + c_2\sin 2t)$

$2 = x(0) = c_1$

$x' = -e^{-t}(c_1\cos 2t + c_2\sin 2t)$

$+ e^{-t}(-2c_1\sin 2t + 2c_2\cos 2t)$

$4\sqrt{3} - 2 = x'(0) = -c_1 + 2c_2$ $\therefore 4\sqrt{3} \cancel{-2} = \cancel{-2} + 2c_2$

$2\sqrt{3} = c_2$

$$x(t) = e^{-t}\left(2\cos 2t + 2\sqrt{3}\sin 2t\right)$$

$$\sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$$

$$x(t) = 4e^{-t}\left(\frac{1}{2}\cos 2t + \frac{\sqrt{3}}{2}\sin 2t\right)$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\boxed{x(t) = 4e^{-t}\cos\left(2t - \frac{\pi}{3}\right)}$$