Math 242 Spring 1994 Ex 1

(1) Consider $y' = f(x,y)$ $y(a) = b$.

(a) If $f(x,y)$ is continuous on some rectangle which contains (a,b) in its interior then the initial value problem has at least one solution on some interval containing $x=a$.

(b) If f_y is continuous on some rectangle which contains (a,b) in its interior, then the initial value problem has exactly one solution on some interval containing $x=a$.

(2) (a) $f(x,y) = \dfrac{(2+y)^2}{1+x^2}$ is continuous everywhere

$f_y(x,y) = \dfrac{2(2+y)}{1+x^2}$ is continuous everywhere

Thus the IVP has a unique solution on some interval containing $x=0$

(b) $\displaystyle\int \frac{dy}{(2+y)^2} = \int \frac{dx}{x^2+1}$ $\qquad \dfrac{-1}{(2+y)} = \arctan x + c$

$\dfrac{-1}{\arctan x + c} - 2 = y$ $\qquad -\dfrac{1}{c} - 2 = y(0) = 0$ $\quad \therefore c = -\frac{1}{2}$

$\dfrac{-1}{\arctan x - \frac{1}{2}} - 2 = y$

(c) The work of (a) shows that the IVP has a unique solution on some interval containing $x=0$.

(d) $y = -2$

(3) $y = \int x \sin x \, dx = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + c$

$u = x$ $\quad v = -\cos x$

$du = dx$ $\quad dv = \sin x \, dx$

$0 = y(0) = c$

$y = -x\cos x + \sin x$

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(4) $y' - \frac{3}{x}y = x^3\cos x \qquad \mu = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}$

$\frac{1}{x^3}y' - \frac{3}{x^4}y = \cos x \qquad \frac{1}{x^3}y = \int \cos x\, dx = \sin x + C$

$y = x^3\sin x + cx^3$

$0 = y(2\pi) = c(2\pi)^3 \qquad y = x^3\sin x$

(5) $A(t)$ = # of lbs of salt in the tank at time t sec.

$\frac{dA}{dt} = \frac{1\text{ lb}}{\text{gal}}\,5\frac{\text{gal}}{\text{sec}} - \frac{A\text{ lb}}{100+2t\text{ gal}}\,3\frac{\text{gal}}{\text{sec}} \qquad A(0) = 50 \qquad \text{Find } A(150)$

$\frac{dA}{dt} + \frac{3}{100+2t}A = 5 \qquad \mu = e^{\int \frac{3}{100+2t}dt} = (100+2t)^{\frac{3}{2}}$

$(100+2t)^{\frac{3}{2}}\frac{dA}{dt} + 3(100+2t)^{\frac{1}{2}}A = 5(100+2t)^{\frac{3}{2}}$

$(100+2t)^{\frac{3}{2}}A = 5(100+2t)^{\frac{5}{2}}\left(\frac{2}{5}\right)\left(\frac{1}{2}\right) + C$

$A = (100+2t) + C(100+2t)^{-\frac{3}{2}}$

$50 = A(0) = 100 + C(100)^{-\frac{3}{2}} \qquad \therefore -50(1000) = C$

$A(t) = (100+2t) - 50{,}000(100+2t)^{-\frac{3}{2}}$

$A(150) = 400 - \frac{50{,}000}{(400)^{\frac{3}{2}}} = 393.75\text{ lb}$

(6) $A(t)$ = # of grams of substance at time t years.

$\frac{dA}{dt} = kA \qquad A(t) = 2e^{kt} \qquad 1.4 = A(1) = 2e^{k} \qquad \ln .7 = k$

$A(0) = 2$

$A(10) = 2e^{10\ln .7} = 2(.7)^{10} \cong .05649\text{ gms.}$

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