1990 Exam I 242

① $2xyy' = x^2 + 2y^2$ Let $v = \frac{y}{x}$ $xv = y$ $xv' + v = y'$

$2y' = \frac{x}{y} + 2\frac{y}{x}$

$2(xv' + v) = \frac{1}{v} + 2v$

$2xv' = \frac{1}{v}$

$2v\,dv = \frac{dx}{x}$

$v^2 = \ln|x| + c$

$$y = \pm x\sqrt{\ln|x| + c}$$

② $y' = \frac{(1+y)^2}{1+x^2}$ so $y' = f(x,y)$ for

$f(x,y) = \frac{(1+y)^2}{1+x^2}$

$\frac{\partial f}{\partial y} = \frac{2(1+y)}{1+x^2}$ which is continuous everywhere.

ⓐ (*) has a unique solution for all a and b.

ⓑ $\frac{dy}{(1+y)^2} = \frac{dx}{1+x^2}$

$\frac{-1}{1+y} = \arctan x + c$

$1 + y = \frac{-1}{\arctan x + c}$

$y = \frac{-1}{\arctan x + c} - 1$

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$y(0) = 0 \qquad 0 = \dfrac{-1}{0 + C} - 1$

$$1 = \frac{-1}{C}$$

$$C = -1$$

$$\boxed{y = \frac{-1}{\arctan x - 1} - 1}$$

(c) $y(0) = -1$

$$-1 = \frac{-1}{\arctan x + c} = -1$$

$\therefore 0 = \dfrac{-1}{\arctan x + c}$ This doesn't make any sense.

On the other hand $\boxed{y = -1}$ is the unique solution to (*)

(3) Let $X(t)$ = # of lbs of salt in the tank at time t

$X(0) = 50$

$$\frac{dX}{dt} = 5 \frac{lb}{gal} \, 1 \frac{gal}{sec} - 3 \frac{gal}{sec} \frac{X}{100 + 2t} \frac{lbs}{gal}$$

we want $X(150)$

$$\frac{dX}{dt} + \frac{3}{100+2t} X = 5$$

$$\mu = e^{\int \frac{3}{100+2t} dt} = e^{\frac{3}{2} \ln 100+2t} = (100+2t)^{\frac{3}{2}}$$

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$$(100+2t)^{\frac{3}{2}}\frac{dx}{dt} + \frac{3}{(100+2t)^{\frac{5}{2}}} X = 5(100+2t)^{\frac{3}{2}}$$

$$(100+2t)^{\frac{3}{2}} X = (100+2t)^{\frac{5}{2}} + C$$

$$X = (100+2t) + C(100+2t)^{-\frac{3}{2}}$$

$$50 = X(0) = 100 + c(100)^{-\frac{3}{2}}$$

$$-50(1000) = C$$

$$X(150) = 400 - \frac{50{,}000}{(400)^{\frac{3}{2}}} = 393.75 \text{ pounds}$$

(4) $y=\sin x$ and $y=\cos x$ are solutions of
$y''+y=0$

so every solution has the form $y = c_1 \sin x + c_2 \cos x$
$y' = c_1 \cos x - c_2 \sin x$

$2 = c_1(0) + c_2(1)$
$3 = c_1(1) - c_2(0)$

$$\boxed{y = 2\sin x + 3\cos x}$$

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