Final S 2021
Math 241

16 ① Let $X(t)$ = # lbs of Pollution in the tank at time t.

$$\boxed{\begin{aligned}\frac{dx}{dt} &= 4\frac{g}{h}\,2\frac{lb}{g} - 2\frac{g}{h}\,\frac{x \text{ lb}}{400+2t}\,\frac{lb}{g} \\ X(0) &= 100\end{aligned}}$$

16 ② $\mathcal{L}(\cos^2 2t) = \mathcal{L}\left(\frac{1}{2}(1+\cos 4t)\right) = \boxed{\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)}$

$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$
$\cos(\theta-\varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$

$\cos(\theta+\varphi)+\cos(\theta-\varphi) = 2\cos\theta\cos\varphi$

$\frac{1}{2}(\cos 2\theta+1) \quad = \cos^2\theta$

17 ③ $y' - y = y^2 e^x$ This is a Bernoulli equation

Let $v = y^{1-2} = y^{-1}$

so $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$

Multiply both sides by $-y^{-2}$

$-y^{-2}y' + y^{-1} = -e^x$

$\frac{dv}{dx} + v = -e^x$

Multiply both sides by $\mu(x) = e^{\int 1\,dx} = e^x$

$\underbrace{\frac{dv}{dx}e^x + e^x v}_{\frac{d}{dx}(e^x v)} = -e^{2x}$

Integrate both sides w.r.t x

$e^x v = -\frac{1}{2}e^{2x} + C$

$v = -\frac{1}{2}e^x + Ce^{-x}$

$$y^{-1} = -\tfrac{1}{2}e^{x} + ce^{-x}$$

$$\boxed{\frac{1}{-\frac{1}{2}e^{x} + ce^{-x}} = y}$$

Check $y' - y = -\left(-\tfrac{1}{2}e^{x} + ce^{-x}\right)^{-2}\left(-\tfrac{1}{2}e^{x} - ce^{-x}\right) - \left(-\tfrac{1}{2}e^{x} + ce^{-x}\right)^{-1}$

$$= \left(-\tfrac{1}{2}e^{x} + ce^{-x}\right)^{-2}\underbrace{\left[\tfrac{1}{2}e^{x} + ce^{-x} - \left(-\tfrac{1}{2}e^{x} + ce^{-x}\right)\right]}_{+e^{x}}$$

$$= y^2 e^{x} \checkmark$$

17 (4) To solve the homogeneous problem $y'' + 4y' + 4y = 0$

We consider the characteristic polynomial $r^2 + 4r + 4 = (r+2)^2$

The solution of the homog. problem is

$$y = c_1 e^{-2x} + c_2 xe^{-2x}$$

We look for a solution of the given DE of the form $y = Ae^{2x}$,

$y = Ae^{2x}$ is a sol of the given equation provided

$$4Ae^{2x} + 8Ae^{2x} + 4Ae^{2x} = e^{2x}$$

This happens when $A = \frac{1}{16}$

The general solution of the given DE is

$$y = c_1 e^{-2x} + c_2 xe^{-2x} + \tfrac{1}{16}e^{2x}$$

17 (5) This is a homogeneous DE

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}$$

Let $v = \frac{y}{x}$

So $xv = y$ $\quad x\frac{dv}{dx} + v = \frac{dy}{dx}$

$$x\frac{dv}{dx} + v = \frac{1}{v} + v$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

$$\int v\,dv = \int \frac{dx}{x}$$

$$\frac{v^2}{2} = \ln|x| + c$$

$$v^2 = 2\ln|x| + K \quad \text{where } K = 2c$$

$$v^2 = \ln x^2 + K$$

$$v = \pm\sqrt{\ln(x^2) + K}$$

$$\frac{y}{x} = \pm\sqrt{\ln(x^2) + K}$$

$$\boxed{y = +x\sqrt{\ln(x^2) + K} \quad \text{or} \quad y = -x\sqrt{\ln(x^2) + K}}$$

We check $y = +x\sqrt{\ln(x^2)+K}$!

Observe that $xy\frac{dy}{dx} = xy\left(\frac{x\,\frac{2x}{x^2}}{2\sqrt{\ln(x^2)+K}} + \sqrt{\ln(x^2)+K}\right)$

$$= x\,x\sqrt{\ln x^2 + K}\left(\frac{1}{\sqrt{\ln(x^2)+K}} + \sqrt{\ln(x^2)+K}\right)$$

$$= x^2 + x^2\left(\sqrt{\ln(x^2)+K}\right)^2$$

$$= x^2 + y^2 \checkmark$$

(6) Let $X = \mathcal{L}(x)$

17 It follows that $\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = sX + 1$

$$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s(sX+1) - 2 = s^2X + s - 2$$

Transform the DE:

$$s^2X + s - 2 - 10(sX+1) + 9X = \frac{5}{s^2}$$

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$$(s^2-10s+9)X + s - 12 = \frac{5}{s^2}$$

$$X = \frac{\frac{5}{s^2} - s + 12}{s^2-10s+9} = \frac{5 - s^3 + 12s^2}{\underbrace{(s^2-10s+9)}_{(s-9)(s-1)} s^2}$$

$$x = \mathcal{L}^{-1}\left(\frac{-s^3+12s^2+5}{(s-9)(s-1)\,s^2}\right)$$

We apply the technique of Partial Fractions

$$\frac{-s^3+12s^2+5}{(s-9)(s-1)s^2} = \frac{A}{s-9} + \frac{B}{s-1} + \frac{C}{s} + \frac{D}{s^2}$$

$$-s^3+12s^2+5 = A(s-1)s^2 + B(s-9)s^2 + C(s-9)(s-1)s + D(s-9)(s-1)$$

Plug in $s = 1$ to learn $-1+12+5 = (-8)B$

$$-2 = B$$

plug in $s=9$ to learn

$$\underbrace{-9^3+12\,9^2}_{9^2(3)}+5 = A\,8(81)$$

$$\frac{31}{81} = \frac{248}{8(81)} = \frac{3(81)+5}{8(81)} = A$$

Plug in $s=0$ to learn

$$5 = 9D$$

$$\frac{5}{9} = D$$

The coeff of s on the left is 0

The coef of s on the right is $9C + D(-10)$

$$\therefore\ 9C - 10D = 0$$

$$C = \frac{10D}{9} = \frac{50}{81}$$

$$x = \mathcal{L}^{-1}\left(\frac{\frac{248}{648}}{s-9} + \frac{-2}{s-1} + \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} \right)$$

$$\boxed{x(t) = \frac{31}{81} e^{9t} - 2e^{t} + \frac{50}{81} + \frac{5}{9} t}$$

check $x'(t) = \frac{31}{9} e^{9t} - 2e^{t} + \frac{5}{9}$

$x''(t) = 31 e^{9t} - 2e^{t}$

$$\begin{aligned} x'' - 10x' + 9x = {} & 31e^{9t} - 2e^{t} \\ & -10\tfrac{31}{9}e^{9t} + 20e^{t} - \tfrac{50}{9} \\ & \tfrac{31}{9}e^{9t} - 18e^{t} + \tfrac{50}{9} + 5t \\ = {} & 5t ✓ \end{aligned}$$

$x(0) = \frac{31}{81} - 2 + \frac{50}{81} = -1$ ✓

$x'(0) = \frac{31}{9} - 2 + \frac{5}{9} = 4 - 2 = 2$ ✓