Quiz 4, February 16, 2017, 11:40 class

Solve $x^2y' + 2xy = 5y^3$. Express your answer in the form $y = y(x)$. Please check your answer.
ANSWER: This is a Bernoulli equation. We let $v = y^{1-3} = y^{-2}$. It follows that $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$. Multiply both sides by $-2y^{-3}$ to obtain:

$$x^2(-2y^{-3}y') - 4xy^{-2} = -10$$
$$x^2\tfrac{dv}{dx} - 4xv = -10.$$

Divide both sides by x^2:

$$\tfrac{dv}{dx} - 4x^{-1}v = -10x^{-2}.$$

Multiply both sides by

$$\mu = e^{-4\int x^{-1}dx} = e^{-4\ln(x)} = x^{-4}$$

to obtain

$$x^{-4}\frac{dv}{dx} - 4x^{-5}v = -10x^{-6}$$

Notice that the left side is

$$\tfrac{d}{dx}(x^{-4}v).$$

Integrate both sides to obtain

$$x^{-4}v = 2x^{-5} + C$$
$$v = 2x^{-1} + Cx^4$$
$$y^{-2} = (2x^{-1} + Cx^4)$$
$$\boxed{y = \frac{1}{\sqrt{2x^{-1} + Cx^4}}}.$$

Check. Plug the proposed answer into the left side of the original DE to obtain

$$x^2y' + 2xy = x^2(\tfrac{-1}{2})(2x^{-1} + Cx^4)^{-3/2}(-2x^{-2} + 4Cx^3) + 2x(2x^{-1} + Cx^4)^{-1/2}$$
$$= (2x^{-1} + Cx^4)^{-3/2}[x^2(\tfrac{-1}{2})(-2x^{-2} + 4Cx^3) + 2x(2x^{-1} + Cx^4)]$$
$$= (2x^{-1} + Cx^4)^{-3/2}[(1 - 2Cx^5) + (4 + 2Cx^5)]$$
$$= (2x^{-1} + Cx^4)^{-3/2}(5) = 5y^3. \checkmark$$