PRINT Your Name:__________________________________

Quiz for February 26, 2013

The quiz is worth 5 points. **Remove EVERYTHING from your desk except this quiz and a pen or pencil.** SHOW your work. Express your work in a neat and coherent manner. $\boxed{\text{BOX}}$ your answer.

1. Consider a population $P(t)$ which satisfies the Differential Equation

$$\frac{dP}{dt} = aP^2 - bP, \tag{1}$$

where a and b are positive constants. Let $B(t) = aP(t)^2$ and $D(t) = bP(t)$. Call $B(t)$ the birth rate at time t and $D(t)$ the death rate at time t. When we first thought about population models we learned that $P(t) = 0$ and $P(t) = M$ are equilibrium solutions for the Differential Equation

$$\frac{dP}{dt} = kP(P - M), \tag{2}$$

where k and M are positive constants. We also learned that the solution of (2) is

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{kMt}}.$$

(a) Express the "M" of (2) in terms of the data $B(0)$, $D(0)$, and $P(0)$ from (1).

(b) Suppose that some population is modeled by (1) and that the initial population is 100 and there are 10 births per month and 9 deaths per month occurring at $t = 0$. How many months does it take until $P(t)$ reaches 10 times the threshold population M?

First we do (a). We write (1) in the form $\frac{dP}{dt} = aP(P - \frac{b}{a})$. We now see that $M = \frac{b}{a}$. On the other hand, when we put $t = 0$ into the equations $B(t) = aP(t)^2$ and $D(t) = bP(t)$, we learn that the constants a and b are $\frac{B(0)}{P(0)^2} = a$ and $\frac{D(0)}{P(0)} = b$; and therefore,

$$M = \frac{b}{a} = \frac{\frac{D(0)}{P(0)}}{\frac{B(0)}{P(0)^2}} = \boxed{\frac{D(0)P(0)}{B(0)}.}$$

Now we do (b). We are told that $P(0) = 100$, $B(0) = 10$, $D(0) = 9$, $M = \frac{D(0)P(0)}{B(0)} = \frac{9 \cdot 100}{10} = 90$, and $kM = b = \frac{D(0)}{P(0)} = \frac{9}{100}$. So we know that

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{kMt}} = \frac{90 \cdot 100}{100 + (90 - 100)e^{\frac{9t}{100}}}.$$

We are supposed to find t with $P(t) = 900$. We solve the following equation for t:

$$
\begin{aligned}
900 &= \frac{90 \cdot 100}{100 + (90 - 100)e^{\frac{9t}{100}}} \\
1 &= \frac{10}{100 + (90 - 100)e^{\frac{9t}{100}}} \\
100 + (90 - 100)e^{\frac{9t}{100}} &= 10 \\
-10e^{\frac{9t}{100}} &= -90 \\
e^{\frac{9t}{100}} &= 9 \\
\frac{9t}{100} &= \ln 9
\end{aligned}
$$

The population hits 900 when the time is $\boxed{\frac{100}{9} \ln 9}$ months.