Math 242, Exam 3, Summer 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **6** problems. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. (7 points) **Solve** $y'' + 2y' - 3y = e^x$ **. Express your answer in the form** $y(x)$ **.**

First we solve the homogeneous problem $y'' + 2y' - 3y = 0$. We try $y = e^{rx}$ and solve $0 = r^2 + 2r - 3 = (r-1)(r+3)$. The general solution of the homogeneous equation $y''+2y'-3y = 0$ is $c_1e^x+c_2e^{-3x}$. Now we find a particular solution of the given non-homogeneous problem. We try $y = Axe^x$. We compute $y' = e^x(Ax+A)$ and $y'' = e^x(Ax + 2A)$. Plug our candidate into the original DE to obtain:

$$e^x = e^x(Ax + 2A) + 2e^x(Ax + A) - 3Axe^x = 4Ae^x.$$

Take $A = 1/4$. The general solution of the given DE is

$$\boxed{y = c_1e^x + c_2e^{-3x} + \tfrac{1}{4}xe^x.}$$

2. (7 points) **Solve** $y''' - 5y'' + 8y' - 4y = 0$ **. Express your answer in the form** $y(x)$ **.**

Try $y = e^{rx}$. We see that

$$0 = r^3 - 5r^2 + 8r - 4 = (r-1)(r^2 - 4r + 4) = (r-1)(r-2)^2.$$

The solution of the DE is

$$\boxed{y = c_1e^x + c_2e^{2x} + c_3xe^{2x}.}$$

3. (6 points) **Solve** $y'' + y' + y = 0$ **. Express your answer in the form** $y(x)$ **.**

Try $y = e^{rx}$. We see that $r^2 + r + 1 = 0$. Thus $r = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm i\sqrt{3}}{2}$. The solution of the DE is

$$\boxed{y = e^{-x/2}(c_1\cos(\tfrac{\sqrt{3}x}{2}) + c_2\sin(\tfrac{\sqrt{3}x}{2})).}$$

4. (6 points) **Suppose that a body moves through a resisting medium with resistance proportional to its velocity $v(t)$, so that $\frac{dv}{dt} = -kv$ for some positive constant k. Let $x(t)$ be the position of the object at time t. Let $v(0) = v_0$ and $x(0) = x_0$. Find the velocity and position of the object at time t. Find $\lim_{t\to\infty} x(t)$.**

Seperate the variables and integrate to see that

$$\frac{dv}{v} = -kdt$$

$$\ln|v| = -kt + C$$

$$v = \pm e^C e^{-kt}.$$

Let K be the constant $\pm e^C$. We have

$$v(t) = Ke^{-kt}.$$

Plug in $t = 0$ to see that $\boxed{v(t) = v_0 e^{-kt}}$. Seperate the variables and integrate to see that

$$\frac{dx}{dt} = v_0 e^{-kt}$$

$$dx = v_0 e^{-kt} dt$$

$$x = \frac{v_0}{-k} e^{-kt} + C_2$$

Plug in $t = 0$ to see that

$$x_0 = x(0) = \frac{v_0}{-k} + C_2;$$

so $x_0 + \frac{v_0}{k} = C_2$ and

$$\boxed{x(t) = \frac{v_0}{-k} e^{-kt} + x_0 + \frac{v_0}{k}}.$$

We compute

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{v_0}{-k} e^{-kt} + x_0 + \frac{v_0}{k} = \boxed{x_0 + \frac{v_0}{k}}.$$

5. (6 points) **Consider a population** $P(t)$ **which satisfies the Differential Equation**

$$\frac{dP}{dt} = aP^2 - bP, \tag{1}$$

where a **and** b **are positive constants. Let** $B(t) = aP(t)^2$ **and** $D(t) = bP(t)$ **. Call** $B(t)$ **the birth rate at time** t **and** $D(t)$ **the death rate at time** t **. When we first thought about population models we learned that** $P(t) = 0$ **and** $P(t) = M$ **are equilibrium solutions for the Differential Equation**

$$\frac{dP}{dt} = kP(P - M), \tag{2}$$

where k **and** M **are positive constants. We also learned that the solution of (2) is**

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{kMt}}.$$

(a) **Express the "** M **" of (2) in terms of the data** $B(0)$ **,** $D(0)$ **, and** $P(0)$ **from (1).**

First we do (a). We write (1) in the form $\frac{dP}{dt} = aP(P - \frac{b}{a})$. We now see that $M = \frac{b}{a}$. On the other hand, when we put $t = 0$ into the equations $B(t) = aP(t)^2$ and $D(t) = bP(t)$, we learn that the constants a and b are $\frac{B(0)}{P(0)^2} = a$ and $\frac{D(0)}{P(0)} = b$; and therefore,

$$M = \frac{b}{a} = \frac{\frac{D(0)}{P(0)}}{\frac{B(0)}{P(0)^2}} = \boxed{\frac{D(0)P(0)}{B(0)}.}$$

(b) **Suppose that some population is modeled by (1) and that the initial population is** 100 **and there are** 10 **births per month and** 9 **deaths per month occurring at** $t = 0$ **. How many months does it take until** $P(t)$ **reaches** 10 **times the threshold population** M **?**

Now we do (b). We are told that $P(0) = 100$, $B(0) = 10$, $D(0) = 9$, $M = \frac{D(0)P(0)}{B(0)} = \frac{9 \cdot 100}{10} = 90$, and $kM = b = \frac{D(0)}{P(0)} = \frac{9}{100}$. So we know that

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{kMt}} = \frac{90 \cdot 100}{100 + (90 - 100)e^{\frac{9t}{100}}}.$$

We are supposed to find t with $P(t) = 900$. We solve the following equation for t:

$$900 = \frac{90 \cdot 100}{100 + (90 - 100)e^{\frac{9t}{100}}}$$

$$1 = \frac{10}{100 + (90 - 100)e^{\frac{9t}{100}}}$$

$$100 + (90 - 100)e^{\frac{9t}{100}} = 10$$

$$-10e^{\frac{9t}{100}} = -90$$

$$e^{\frac{9t}{100}} = 9$$

$$\frac{9t}{100} = \ln 9$$

The population hits 900 when the time is $\boxed{\frac{100}{9}\ln 9}$ months.

6. (6 points) **Consider the initial value problem** $\frac{dy}{dx} = x + y^3$ **,** $y(1) = 2$ **. Use Euler's method to approximate** $y(3/2)$ **. Use two steps, each of size** $1/4$ **.**

Consider the picture:

The curve represents the correct solution of the initial value problem. We approximate the real solution with the two line segments. The line segment from $(1,2)$ to $(\frac{5}{4}, y_1)$ has slope equal to $(x + y^3)|_{(1,2)}$. The line segment from $(\frac{5}{4}, y_1)$ to $(\frac{3}{2}, y_2)$ has slope equal to $(x + y^3)|_{(\frac{5}{4}, y_1)}$. Of course, y_2 is our approximation of $y(\frac{3}{2})$.

We see that $(x+y^3)|_{(1,2)} = 9$; so, the segment from $(1,2)$ to $(\frac{5}{4}, y_1)$ has slope equal to 9. This segment lives on the line

$$y - 2 = 9(x - 1),$$

which is $y = 9x - 7$. Thus, $y_1 = 45/4 - 7 = 17/4$.

The line segment from $(\frac{5}{4}, y_1)$ to $(3/2, y_2)$ has slope equal to

$$(x+y^3)|_{(\frac{5}{4},y_1)} = (x+y^3)|_{(\frac{5}{4},\frac{17}{4})} = \frac{5}{4} + \left(\frac{17}{4}\right)^3 = \frac{17^3+16(5)}{64}.$$

This segment lives on the line

$$y - \frac{17}{4} = \frac{17^3+16(5)}{64}\left(x - \frac{5}{4}\right)$$

which is

$$y = \frac{17^3+16(5)}{64}\left(x - \frac{5}{4}\right) + \frac{17}{4}$$

Our approximation of $y(3/2)$ is

$$\boxed{y_2 = \frac{17^3+16(5)}{64}\left(\frac{1}{4}\right) + \frac{17}{4}.}$$