Math 242, Exam 3, Solution, Fall 2012

You should KEEP this piece of paper. Write everything on the blank paper provided. If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

Nothing may be on your desk except things that came from me. In particular, **no Calculators or Cell phones** may be on your desk.

Your work must be coherent and correct.

The solutions will be posted later today.

1. (13 points) **Find the general solution of** $y'' + 2y' + y = xe^{-x}$. **Express your answer in the form** $y(x)$**. Check your answer.**

We solve the homogeneous problem by trying $y = e^{rx}$ and getting $r^2 + 2r + 1 = 0$ and this is the same as $(r+1)^2 = 0$. So the general solution of the homogeneous problem is $y = c_1e^{-x} + c_2xe^{-x}$. We would like to look for solutions of the given problem of the form $Axe^{-x} + Be^{-x}$. However, it is clear that none of these functions satisfy the given non-homogeneous problem because they all satisfy the homogeneous problem; so instead we look for solutions of the given non-homogeneous problem of the form $y = e^{-x}(Ax^3 + Bx^2)$. We plug our candidate into the given non-homogeneous DE and determine the coefficients A and B that work:

$$y' = e^{-x}(3Ax^2 + 2Bx) - e^{-x}(Ax^3 + Bx^2) = e^{-x}(-Ax^3 + (3A - B)x^2 + 2Bx)$$

$$\begin{aligned} y'' &= e^{-x}(-3Ax^2 + 2(3A - B)x + 2B) - e^{-x}(-Ax^3 + (3A - B)x^2 + 2Bx) \\ &= e^{-x}(Ax^3 + (-6A + B)x^2 + (6A - 4B)x + 2B). \end{aligned}$$

The DE becomes

$$xe^{-x} = \begin{cases} \quad e^{-x}(Ax^3 + (-6A + B)x^2 + (6A - 4B)x + 2B) \\ +2\big(e^{-x}(-Ax^3 + (3A - B)x^2 + 2Bx)\big) \\ + \quad e^{-x}(Ax^3 + Bx^2). \end{cases}$$

We solve $xe^{-x} = e^{-x}(6Ax + 2B)$. So $A = \frac{1}{6}$ and $B = 0$. The general solution of the original DE is

$$\boxed{y = c_1e^{-x} + c_2xe^{-x} + \tfrac{1}{6}x^3e^{-x}.}$$

Check. We plug $y = e^{-x}(c_1 + c_2x + \frac{1}{6}x^3)$ into the DE. We see that

$$y' = e^{-x}(c_2 + \tfrac{1}{2}x^2) - e^{-x}(c_1 + c_2x + \tfrac{1}{6}x^3) = e^{-x}(-c_1 + c_2 - c_2x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3)$$

$$\begin{aligned} y'' &= e^{-x}(-c_2 + x - \tfrac{1}{2}x^2) - e^{-x}(-c_1 + c_2 - c_2x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3) \\ &= e^{-x}(c_1 - 2c_2 + x + c_2x - x^2 + \tfrac{1}{6}x^3). \end{aligned}$$

Thus $y'' + 2y' + y$ is equal to

$$\left\{ \begin{array}{cccccc} e^{-x}(& c_1 - 2c_2 & +(1+c_2)x & -x^2 & +\frac{1}{6}x^3) \\ +2e^{-x}(& -c_1 + c_2 & -c_2x & +\frac{1}{2}x^2 & -\frac{1}{6}x^3) \\ +e^{-x}(& c_1 & +c_2x & & +\frac{1}{6}x^3), \end{array} \right.$$

and this is xe^{-x} as desired.

2. (13 points) **The Initial Value Problem**

$$x'' + 2x' + 5x = 0, \quad x(0) = 2, \quad x'(0) = 4\sqrt{3} - 2$$

describes the motion of a spring. Solve the problem and put your solution in the form

$$x(t) = Ce^{-pt}\cos(\omega t - \alpha).$$

Check your answer.

We try $x = e^{rt}$. Plug our proposed solution into the Differential Equation to see that $r^2 + 2r + 5 = 0$ (since e^{rt} is never zero). Use the quadratic formula to see that $r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$. The corresponding solutions to the DE are $x = e^{-t}\cos 2t$ and $x = e^{-t}\sin 2t$. Indeed, the general solution of the DE is $x = e^{-t}(c_1\cos 2t + c_2\sin 2t)$. We find c_1 and c_2 so that the Initial Conditions are satisfied.

$$x' = e^{-t}(-2c_1\sin 2t + 2c_2\cos 2t) - e^{-t}(c_1\cos 2t + c_2\sin 2t)$$

$$= e^{-t}((-2c_1 - c_2)\sin 2t + (2c_2 - c_1)\cos 2t).$$

We solve

$$2 = x(0) = c_1 \quad \text{and} \quad 4\sqrt{3} - 2 = x'(0) = 2c_2 - c_1.$$

Thus, $c_1 = 2$ and $c_2 = 2\sqrt{3}$. The solution of our IVP is

$$x = e^{-t}(2\cos 2t + 2\sqrt{3}\sin 2t) = 4e^{-t}(\tfrac{1}{2}\cos 2t + \tfrac{\sqrt{3}}{2}\sin 2t)$$

$$= 4e^{-t}(\cos\tfrac{\pi}{3}\cos 2t + \sin\tfrac{\pi}{3}\sin 2t) = 4e^{-t}\cos(2t - \tfrac{\pi}{3}).$$

Our solution is $\boxed{x(t) = 4e^{-t}\cos(2t - \tfrac{\pi}{3}).}$

Check. We compute

$$x'(t) = -8e^{-t}\sin(2t - \tfrac{\pi}{3}) - 4e^{-t}\cos(2t - \tfrac{\pi}{3})$$

$$x''(t) = -16e^{-t}\cos(2t - \tfrac{\pi}{3}) + 8e^{-t}\sin(2t - \tfrac{\pi}{3}) + 8e^{-t}\sin(2t - \tfrac{\pi}{3}) + 4e^{-t}\cos(2t - \tfrac{\pi}{3})$$

$$= -12e^{-t}\cos(2t - \tfrac{\pi}{3}) + 16e^{-t}\sin(2t - \tfrac{\pi}{3}).$$

We see that

$$x'' + 2x' + 5x = \begin{cases} -12e^{-t}\cos(2t - \frac{\pi}{3}) + 16e^{-t}\sin(2t - \frac{\pi}{3}) \\ +2\left(-4e^{-t}\cos(2t - \frac{\pi}{3}) - 8e^{-t}\sin(2t - \frac{\pi}{3})\right) \\ +5\left(4e^{-t}\cos(2t - \frac{\pi}{3})\right), \end{cases}$$

and this is zero. We also see that $x(0) = 4\cos(-\frac{\pi}{3}) = 4(\frac{1}{2}) = 2$ and $x'(0) = -8\sin(-\frac{\pi}{3}) - 4\cos(-\frac{\pi}{3}) = -8(-\frac{\sqrt{3}}{2}) - 4(\frac{1}{2}) = 4\sqrt{3} - 2$.

3. (12 points) **Find the general solution of $xy' = y + 2\sqrt{xy}$. Express your answer in the form $y(x)$. Check your answer.**

This is a homogeneous problem. Let $v = \frac{y}{x}$. It follows that $xv = y$ and $x\frac{dv}{dx} + v = \frac{dy}{dx}$. Divide both sides by x . Keep in mind that $x = \sqrt{x^2}$ (at least when x is positive). The problem is $y' = \frac{y}{x} + 2\sqrt{\frac{y}{x}}$ and this problem becomes $x\frac{dv}{dx} + v = v + 2\sqrt{v}$. The problem now is $x\frac{dv}{dx} = 2\sqrt{v}$. Separate the variables and integrate:

$$\int v^{-1/2}dv = \int \frac{2}{x}dx$$

$$2v^{1/2} = 2\ln|x| + C$$

$$v^{1/2} = \ln|x| + K$$

(where $K = C/2$)

$$v = (\ln|x| + K)^2$$

$$\frac{y}{x} = (\ln|x| + K)^2$$

$$\boxed{y = x(\ln|x| + K)^2}$$

Check. Take x to be positive. Plug our proposed solution into the DE. The LHS becomes

$$x(2x(\ln x + K)\tfrac{1}{x} + (\ln x + K)^2) = x(2(\ln x + K) + (\ln x + K)^2)$$

$$= 2x(\ln x + K) + x(\ln x + K)^2 = 2\sqrt{xy} + y. \checkmark$$

4. (12 points) **Find the general solution of** $x\frac{dy}{dx}+6y=3xy^{4/3}$ **. Express your answer in the form** $y(x)$ **. Check your answer.**

This is a Bernoulli Equation. Let $v=y^{1-4/3}=y^{-1/3}$. We claculate $\frac{dv}{dx}=-\frac{1}{3}y^{-4/3}\frac{dy}{dx}$. Multiply the DE by $y^{-4/3}$ to obtain $xy^{-4/3}\frac{dy}{dx}+6y^{-1/3}=3x$ and this is $-3x\frac{dv}{dx}+6v=3x$. Divide by $-3x$ to obtain: $\frac{dv}{dx}+\frac{-2}{x}v=-1$. Multiply both sides by $\mu(x)=e^{\int\frac{-2}{x}dx}=e^{-2\ln x}=x^{-2}$ to get $x^{-2}\frac{dv}{dx}-2x^{-3}v=-x^{-2}$. Integrate both sides with respect to x :

$$x^{-2}v=x^{-1}+C.$$

Multiply by x^2 and replace v with $y^{-1/3}$. We have:

$$\boxed{y=(x+Cx^2)^{-3}}$$

Check. Plug our proposed solution into the DE. The LHS becomes

$$-3(1+2Cx)(x+Cx^2)^{-4}x+6(x+Cx^2)^{-3}=(x+Cx^2)^{-4}(-3x(1+2Cx)+6(x+Cx^2))$$

$$=(x+Cx^2)^{-4}3x=3xy^{4/3}.\ \checkmark$$