Math 242, Exam 3, Spring, 2017 11:40 class

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today. The exams will be returned in class on Thursday, March 30.

No Calculators or Cell phones.

(1) **Consider the initial value problem** $\frac{dy}{dx} = y^2 + \frac{1}{x}$**,** $y(3) = 1$**. Use Euler's method to approximate** $y(32/10)$**. Use two steps, each of size** $1/10$**.**

Let $f(x, y) = y^2 + \frac{1}{x}$, $(x_0, y_0) = (3, 1)$, $x_1 = \frac{31}{10}$, and $x_2 = \frac{32}{10}$. Define y_1 so that the slope of the line joining (x_0, y_0) to (x_1, y_1) is $f(x_0, y_0)$. Define y_2 so that the slope of the line joining (x_1, y_1) to (x_2, y_2) is $f(x_1, y_1)$. Then y_2 is our approximation of $y(\frac{32}{10})$.

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$$

$$\frac{y_1 - 1}{\frac{1}{10}} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$y_1 = 1 + \left(\frac{1}{10}\right)\left(\frac{4}{3}\right) = 1 + \frac{4}{30} = 1 + \frac{2}{15} = \frac{17}{15}.$$

$$\frac{y_2 - y_1}{x_2 - x_1} = f(x_1, y_1)$$

$$\frac{y_2 - \frac{17}{15}}{\frac{1}{10}} = \left(\frac{17}{15}\right)^2 + \frac{10}{31}.$$

$$y_2 = \left(\frac{17}{15}\right)^2 + \frac{1}{10}\left(\left(\frac{17}{15}\right)^2 + \frac{10}{31}\right).$$

$$\boxed{\text{Our approximation of } y(\tfrac{32}{10}) \text{ is } y_2 = \tfrac{17}{15} + \tfrac{1}{10}\left(\left(\tfrac{17}{15}\right)^2 + \tfrac{10}{31}\right).}$$

(2) **Find the general solution of** $y'' + 6y' + 9y = 0$**.**

The characteristic polynomial is $r^2 + 6r + 9 = (r + 3)^2$. The general solution of the differential equation is

$$\boxed{y = c_1 e^{-3x} + c_2 x e^{-3x}}.$$

(3) **Find the general solution of $y'' + 3y' + 2y = e^x$.**

We first solve the homogeneous problem. The characteristic polynomial is $r^2 + 3r + 2 = (r+2)(r+1)$. The solution of the homogeneous problem is $y = c_1 e^{-2x} + c_2 e^{-x}$. Now we look for a particular solution to the given differential equation. We try $y = Ae^x$; We plug this y into the original DE:

$$e^x(A + 3A + 2A) = e^x.$$

So we take $A = 1/6$. The general solution of the DE is

$$\boxed{y = c_1 e^{-2x} + c_2 e^{-x} + (1/6)e^x.}$$

<u>Check:</u> We compute

$$y' = -2c_1 e^{-2x} - c_2 e^{-x} + (1/6)e^x$$
$$y'' = 4c_1 e^{-2x} + c_2 e^{-x} + (1/6)e^x$$

Plug this into the LHS of the DE:

$$(\underline{4c_1 e^{-2x}} + \underline{\underline{c_2 e^{-x}}} + (1/6)e^x) + 3(\underline{-2c_1 e^{-2x}} - \underline{\underline{c_2 e^{-x}}} + (1/6)e^x) + 2(\underline{c_1 e^{-2x}} + \underline{\underline{c_2 e^{-x}}} + (1/6)e^x)$$
$$= e^x. \checkmark$$

(4) **Find the general solution of $\frac{dy}{dx} - \frac{1}{x}y = xy^2$.**

This is a Bernoulli equation. Let $v = y^{-1}$; so $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. Multiply both sides by $-y^{-2}$ to obtain

$$-y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = -x$$
$$\frac{dv}{dx} + \frac{1}{x}v = -x.$$

This is a first order linear problem. Multiply both sides by $e^{\int \frac{1}{x}dx} = x$ to obtain

$$x\frac{dv}{dx} + v = -x^2$$
$$\frac{d}{dx}(xv) = -x^2$$
$$xv = -x^3/3 + C$$
$$v = -x^2/3 + C/x$$
$$1/y = -x^2/3 + C/x$$
$$\boxed{1/(-x^2/3 + C/x) = y}.$$

<u>Check</u>

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{-(-2x/3 - C/x^2)}{(-x^2/3 + C/x)^2} - \frac{1}{x}\frac{1}{(-x^2/3 + C/x)}$$
$$= \frac{1}{(-x^2/3 + C/x)^2}[2x/3 + C/x^2 - \frac{1}{x}(-x^2/3 + C/x)]$$
$$= \frac{1}{(-x^2/3 + C/x)^2}[x] = xy^2. \checkmark$$

(5) **Solve the initial value problem** $y'' - y' - 2y = 8e^{3x}$, $\quad y(0) = -1$, $\quad y'(0) = 11$.

To solve the homogeneous problem, we consider $r^2 - r - 2r = 0$. This is $(r-2)(r+1) = 0$. So, the general solution of the homogeneous problem is $y = c_1 e^{-x} + c_2 e^{2x}$. We look for a number A with $y = Ae^{3x}$ a solution of the given problem. So $9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = 8e^{3x}$. So, $4A = 8$ and $A = 2$. The general solution of the given problem is

$$y = c_1 e^{-x} + c_2 e^{2x} + 2e^{3x}.$$

We compute

$$y' = -c_1 e^{-x} + 2c_2 e^{2x} + 6e^{3x}.$$

We solve

$$\begin{cases} -1 = y(0) = c_1 + c_2 + 2 \\ 11 = y'(0) = -c_1 + 2c_2 + 6 \end{cases}$$

$$\begin{cases} -3 = c_1 + c_2 \\ 5 = -c_1 + 2c_2 \end{cases}$$

$$\begin{cases} 2 = 3c_2 \\ 5 = -c_1 + 2c_2 \end{cases}$$

$$\begin{cases} 2/3 = c_2 \\ 5 = -c_1 + 2(2/3) \end{cases}$$

$$\begin{cases} 2/3 = c_2 \\ c_1 = -5 + 2(2/3) = -11/3 \end{cases}$$

$$\boxed{y = -(11/3)e^{-x} + (2/3)e^{2x} + 2e^{3x}}.$$

<u>Check</u> We compute

$$y = -(11/3)e^{-x} + (2/3)e^{2x} + 2e^{3x}$$

$$y' = (11/3)e^{-x} + (4/3)e^{2x} + 6e^{3x}$$

$$y'' = -(11/3)e^{-x} + (8/3)e^{2x} + 18e^{3x}.$$

So, $y(0) = -(11/3) + (2/3) + 2 = -1$. ✓, $y'(0) = (11/3) + (4/3) + 6 = 11$. ✓

Plug y into the Differential Equation to obtain

$$(-(11/3)e^{-x} + (8/3)e^{2x} + 18e^{3x}) - ((11/3)e^{-x} + (4/3)e^{2x} + 6e^{3x}) - 2(-(11/3)e^{-x} + (2/3)e^{2x} + 2e^{3x})$$

$$= (18 - 6 - 4)e^{3x} = 8e^{3x}. \checkmark$$