Math 242, Exam 2, Summer 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Solve** $(x+y)\frac{dy}{dx} = x - y$ **. Express your answer in the form** $y(x)$ **. Check your answer.**

This is a homogeneous problem. Divide both sides of the equation by x to get $(1+\frac{y}{x})\frac{dy}{dx} = 1 - \frac{y}{x}$. Let $v = \frac{y}{x}$. It follows that $xv = y$ and $x\frac{dv}{dx} + v = \frac{dy}{dx}$. We must solve: $(1+v)(x\frac{dv}{dx} + v) = 1 - v$. Subtract $(1+v)v$ from both sides to get: $(1+v)x\frac{dv}{dx} = 1 - v - (1+v)v$. This is $(1+v)x\frac{dv}{dx} = 1 - 2v - v^2$, which becomes $\frac{1+v}{1-2v-v^2}dv = \frac{dx}{x}$. Integrate both sides. Let $u = 1 - 2v - v^2$. It follows that $du = (-2-2v)dv$. In other words, $du = -2(1+v)dv$. We solve $\frac{-1}{2}\int \frac{du}{u} = \ln|x| + C$; and this is $\ln|u| = -2\ln|x| - 2C$. Exponentiate to obtain $|u| = \frac{e^{-2C}}{x^2}$ or $u = \frac{\pm e^{-2C}}{x^2}$. Let $K = \pm e^{-2C}$. We have $1 - 2v - v^2 = \frac{K}{x^2}$ and this is $1 - 2\frac{y}{x} - (\frac{y}{x})^2 = \frac{K}{x^2}$. Multiply both sides of the equation by x^2 to obtain $x^2 - 2xy - y^2 = K$. To comply with the instructions, you should solve $y^2 + 2xy - x^2 - K = 0$ for y . Use the quadratic formula to see that

$$y = \frac{-2x \pm \sqrt{4x^2 - 4(-x^2 - K)}}{2}.$$

This may be cleaned up to become

$$y = \frac{-2x \pm 2\sqrt{2x^2 + K}}{2};$$

which is

$$\boxed{y = -x \pm \sqrt{2x^2 + K}.}$$

We check $y = -x + \sqrt{2x^2 + K}$. Plug our proposed solution into the DE. The LHS becomes

$$(x+y)\frac{dy}{dx} = (x - x + \sqrt{2x^2+K})(-1 + \frac{4x}{2\sqrt{2x^2+K}}) = \sqrt{2x^2+K}(-1 + \frac{2x}{\sqrt{2x^2+K}})$$

$$= -\sqrt{2x^2+K} + 2x = x - (-x + \sqrt{2x^2+K}) = x - y. \checkmark$$

2. **Solve** $\frac{dy}{dx} = (4x+y)^2$ **. Express your answer in the form** $y(x)$ **. Check your answer.**

This problem is a linear substitution. Let $v = 4x+y$. It follows that $\frac{dv}{dx} = 4+\frac{dy}{dx}$. The problem becomes $\frac{dv}{dx} - 4 = v^2$. Add 4 to both sides and divide by v^2+4 . The DE is $\frac{dv}{v^2+4} = dx$. Integrate to learn that $\frac{1}{2}\arctan(\frac{v}{2}) = x + c$. Multiply by 2 . Take the tangent of each side to obtain $\frac{v}{2} = \tan(2x+2C)$ or $v = 2\tan(2x+2C)$. Of course, $v = 4x+y$. We have found that $\boxed{y = -4x + 2\tan(2x+K)}$, where $K = 2C$.

We check. Plug our proposed solution into the DE. The LHS becomes

$$-4 + 4\sec^2(2x+K) = 4\tan^2(2x+K) = (y+4x)^2. \checkmark$$

3. **Solve** $x\frac{dy}{dx} + 6y = 3xy^{4/3}$ **. Express your answer in the form** $y(x)$ **. Check your answer.**

This is a Bernoulli Equation. Let $v = y^{1-4/3} = y^{-1/3}$. We claculate $\frac{dv}{dx} = -\frac{1}{3}y^{-4/3}\frac{dy}{dx}$. Multiply the DE by $y^{-4/3}$ to obtain $xy^{-4/3}\frac{dy}{dx} + 6y^{-1/3} = 3x$ and this is $-3x\frac{dv}{dx} + 6v = 3x$. Divide by $-3x$ to obtain: $\frac{dv}{dx} + \frac{-2}{x}v = -1$. Multiply both sides by $\mu(x) = e^{\int \frac{-2}{x}dx} = e^{-2\ln x} = x^{-2}$ to get $x^{-2}\frac{dv}{dx} - 2x^{-3}v = -x^{-2}$. Integrate both sides with respect to x :

$$x^{-2}v = x^{-1} + C.$$

Multiply by x^2 and replace v with $y^{-1/3}$. We have:

$$\boxed{y = (x + Cx^2)^{-3}}$$

We check. Plug our proposed solution into the DE. The LHS becomes

$$-3(1+2Cx)(x+Cx^2)^{-4}x + 6(x+Cx^2)^{-3} = (x+Cx^2)^{-4}(-3x(1+2Cx)+6(x+Cx^2))$$

$$= (x+Cx^2)^{-4}3x = 3xy^{4/3}. \checkmark$$

4. **Consider two tanks. The first tank has a volume of 100 gals. of brine. The second tank has a volume of 200 gals. of brine. Each tank initially contains 50 lbs. of salt. Pure water flows into the first tank at the rate of 5 gal./min. The well mixed solution flows out of tank 1 and into tank 2 at the rate of 5 gal./min. The well mixed solution flows out of tank 2 at the rate of 5 gal./min.**

   (a) **How much salt is in the first tank at time** t **?**

Let $x(t)$ be the number of pounds of salt in tank 1 at time t. We know that $\frac{dx}{dt}$ is the rate in minus the rate out. We also know that the rate in is 0 and the rate out is $\frac{x\text{lbs}}{100\text{gal}} \times \frac{5\text{gal}}{\text{min}}$. The DE for x is $\frac{dx}{dt} = -x/20$. Separate the variables, integrate, evaluate the constant: $\boxed{x(t) = 50e^{-t/20}}$.

(b) **How much salt is in the second tank at time t?**

Let $y(t)$ be the number of pounds of salt in tank 2 at time t. We know that $\frac{dy}{dt}$ is the rate in minus the rate out. We also know that the rate in is $\frac{x\text{lbs}}{100\text{gal}} \times \frac{5\text{gal}}{\text{min}} = \frac{5e^{-t/20}}{2}$ and the rate out is $\frac{y\text{lbs}}{200\text{gal}} \times \frac{5\text{gal}}{\text{min}} = y/40$. The DE for y is $\frac{dy}{dt} = \frac{5e^{-t/20}}{2} - y/40$. Write the problem as $\frac{dy}{dt} + y/40 = \frac{5e^{-t/20}}{2}$. Multiply both sides of the equation by $\mu(t) = e^{\int 1/40 dt} = e^{t/40}$ to obtain: $e^{t/40}\frac{dy}{dt} + e^{t/40}y/40 = e^{t/40}\frac{5e^{-t/20}}{2}$. This is

$$\frac{d}{dt}(e^{t/40}y) = \frac{5}{2}e^{-t/40}.$$

Integrate both sides with respect to t to obtain

$$e^{t/40}y = (-40)\frac{5}{2}e^{-t/40} + C.$$

Plug in $t = 0$ to learn $50 = -100 + C$; so $150 = C$ and

$$\boxed{y = -100e^{-t/20} + 150e^{-t/40}}.$$

Of course, we check that $y(0) = -100 + 150 = 50$ and that the proposed solution actually satisfies the DE. The left side becomes:

$$\frac{dy}{dt} = 5e^{-t/20} - \frac{15}{4}e^{-t/40}.$$

The right side becomes:

$$\frac{5e^{-t/20}}{2} - y/40 = \frac{5e^{-t/20}}{2} - (-100e^{-t/20} + 150e^{-t/40})/40$$

$$= \frac{5e^{-t/20}}{2} + \frac{5e^{-t/20}}{2} - 150e^{-t/40}/40.$$

The two sides are equal.

5. **Consider the Differential Equation** $\frac{dx}{dt} = -(3-x)^2$ **.**

(a) **Find all equilibrium solutions** $x(t) = x_e$ **for all** t **for some constant** x_e **.**

Solve $-(3-x)^2 = 0$ to see that $x = 3$ is the only equilibrium solution for the given DE.

(b) **For each equilibrium solution** $x(t) = x_e$ **of the DE, answer the following questions:**

Draw the phase diagram:

(i) **If** $x(0)$ **is a little less than** x_e **, does the corresponding solution** $x(t)$ **head toward or away from the equilibrium solution** $x = x_e$ **.**

If $x(0)$ is a little less than 3, then we look at the phase diagram to see that $x' < 0$, so $x(t)$ is a decreasing function. Thus, $x(t)$ heads away from 3.

(ii) **If** $x(0)$ **is a little more than** x_e **, does the corresponding solution** $x(t)$ **head toward or away from the equilibrium solution** $x = x_e$ **.**

If $x(0)$ is a little more than 3, then we look at the phase diagram to see that $x' < 0$, so $x(t)$ is a decreasing function. Thus, $x(t)$ heads toward 3.

(c) **Sketch a few solutions of the DE.**

(d) **Solve the DE.**

Separate the variables and integrate to see that $\int \frac{dx}{(3-x)^2} dx = -\int dt$. So, $\frac{1}{3-x} = -t + C$. We solve for x: $\frac{1}{-t+C} = 3 - x$; or $x = 3 - \frac{1}{-t+C}$; or $\boxed{x = 3 + \frac{1}{t-C}.}$ By the way, this answer checks. We compute $\frac{dx}{dt} = -\frac{1}{(t-C)^2}$. On the other hand, $-(3-x)^2 = -\left(-\frac{1}{t-C}\right)^2$ and these are the same. Of course the hyperbolas $x = 3 + \frac{1}{t-C}$ look like the pictures given below and this agrees with our expectations.