Math 242, Exam 2, Spring, 2021 Solutions

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, send a picture of your solutions to

kustin@math.sc.edu

ALSO, LEAVE A PHYSICAL COPY OF YOUR SOLUTIONS WITH ME. Fold your solutions in half and write your name on the outisde.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well mixed solution leaves the tank at 3 gal/hr as well. Give an Initial Value Problem for the number of ounces of pollution in the tank at time t hours. Do not solve the Initial Value Problem.**

Let $x(t)$ equal the number of ounces of pollution in the tank at time t hours. We are told that $x(0) = 2$. It is clear that $\frac{dx}{dt}$ is the rate pollution enters the tank minus the rate at which pollution leaves the tank. We are told that polution enters the tank at

$$\left(3\frac{\text{gal}}{\text{hr}}\right)\left(5\frac{\text{ounces}}{\text{gal}}\right).$$

We calculate that pollution leaves the tank at the rate

$$\left(3\frac{\text{gal}}{\text{hr}}\right)\left(\frac{x\text{ounces}}{800\text{gal}}\right).$$

The initial value problem that describes the number of ounces of pollution in the tank at time t hours is

$$\boxed{\frac{dx}{dt} = 15 - \frac{3x}{800} \quad \text{and} \quad x(0) = 2.}$$

(2) **Solve the Initial Value Problem:**

$$\frac{dy}{dx} - \frac{2y}{x} + x^2y^2 = 0 \qquad \textbf{and} \qquad y(1) = \frac{5}{11}.$$

Put your answer in the from $y = y(x)$. Please check your answer.

This is a Bernoulli equation. Let $v = y^{-1}$. Observe that $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. Multiply both sides of the Differential Equation by $-y^{-2}$ to obtain

$$-y^{-2}\frac{dy}{dx} + \frac{1}{y}\frac{2}{x} - x^2 = 0,$$

which is the same as

$$\frac{dv}{dx} + \frac{2}{x}v = x^2.$$

This is a first order linear Differential Equation of the form

$$\frac{dv}{dx} + P(x)v = Q(x).$$

Multiply both sides of the equation by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

to obtain

$$x^2\frac{dv}{dx} + 2xv = x^4.$$

The left side of the equation is the derivative (with respect to x) of x^2v. Integrate both sides with respect to x to obtain

$$x^2v = \frac{x^5}{5} + C$$

Of course $v = \frac{1}{y}$ and $\frac{x^5}{5} + C = \frac{x^5+K}{5}$, where K is the constant $5C$. So, the most recent equation is

$$\frac{x^2}{y} = \frac{x^5 + K}{5}$$

Solve for y:

$$\frac{5x^2}{x^5 + K} = y.$$

Plug in $x = 1$ to see that

$$\frac{5}{1 + K} = \frac{5}{11}.$$

Thus,

$$11 = 1 + K$$

and $K = 10$. The solution to the Initial Value Problem is

$$\boxed{y = \frac{5x^2}{x^5 + 10}.}$$

Check:

$$\frac{dy}{dx} = -\frac{(5x^2)(5x^4)}{(x^5+10)^2} + \frac{10x}{x^5+10} = \frac{-25x^6 + 10x(x^5+10)}{(x^5+10)^2} = \frac{-15x^6 + 100x}{(x^5+10)^2}$$

and

$$\frac{dy}{dx} - \frac{2y}{x} + x^2y^2 = \frac{-15x^6 + 100x}{(x^5+10)^2} - \frac{2}{x}\frac{5x^2}{x^5+10} + x^2\left(\frac{5x^2}{x^5+10}\right)^2$$

$$= \frac{-15x^6 + 100x - 10x(x^5+10) + x^2(25x^4)}{(x^5+10)^2} = 0.\checkmark$$

Also $y(1) = \frac{5}{1+10} = \frac{5}{11}\checkmark$.

(3) **Solve the Initial Value Problem:**

$$xy\frac{dy}{dx} + 4x^2 + y^2 = 0 \qquad \textbf{and} \qquad y(2) = -7.$$

Put your answer in the form $y = y(x)$**. Please check your answer.**

Every term has degree two in x and y. We make the homogeneous substitution $v = \frac{y}{x}$. Divide both sides of the equation by x^2 to obtain

$$\frac{y}{x}\frac{dy}{dx} + 4 + \left(\frac{y}{x}\right)^2 = 0.$$

Let $v = \frac{y}{x}$. It follows that $xv = y$ and $x\frac{dv}{dx} + v = \frac{dy}{dx}$. We solve

$$v\left(x\frac{dv}{dx} + v\right) + 4 + v^2 = 0$$

by separating the variables and integrating.

$$vx\frac{dv}{dx} + v^2 + 4 + v^2 = 0$$

$$vx\frac{dv}{dx} = -(2v^2+4)$$

$$\frac{v}{2v^2+4} = -\frac{1}{x}dx$$

$$\frac{1}{4}\ln(2v^2+4) = -\ln|x| + c$$

$$\ln(2v^2+4) = -4\ln|x| + C$$

(where $C = 4c$)

$$2v^2 + 4 = \frac{e^C}{x^4}$$

$$v^2 + 2 = \frac{\frac{e^C}{2}}{x^4}$$

Let $K = \frac{e^C}{2}$.

$$\left(\frac{y}{x}\right)^2 = \frac{K}{x^4} - 2$$

$$\frac{y}{x} = \pm\sqrt{\frac{K}{x^4} - 2}$$

$$y = \pm x\sqrt{\frac{K}{x^4} - 2}$$

Plug in $x = 2$

$$-7 = \pm 2\sqrt{\frac{K}{16} - 2}$$

Notice that $\pm$ must be $-$ and

$$49 = 4\left(\frac{K}{16} - 2\right)$$

So $K = 228$ and $\boxed{y = -x\sqrt{\frac{228}{x^4} - 2}}$.

Check: Observe that

$$\frac{dy}{dx} = -\frac{x}{2\sqrt{\frac{228}{x^4} - 2}}(-4(228)x^{-5}) - \sqrt{\frac{228}{x^4} - 2}$$

$$= \frac{1}{\sqrt{\frac{228}{x^4} - 2}}\left(-\frac{x}{2}(-4)(228)\frac{1}{x^5} - \left(\frac{228}{x^4} - 2\right)\right)$$

$$= \frac{1}{\sqrt{\frac{228}{x^4} - 2}}\left(\frac{2(228)}{x^4} - \left(\frac{228}{x^4} - 2\right)\right)$$

$$= \frac{\frac{228}{x^4} + 2}{\sqrt{\frac{228}{x^4} - 2}}$$

and

$$xy\frac{dy}{dx} + 4x^2 + y^2 = x\left(-x\sqrt{\frac{228}{x^4} - 2}\right)\frac{\frac{228}{x^4} + 2}{\sqrt{\frac{228}{x^4} - 2}} + 4x^2 + x^2\left(\frac{228}{x^4} - 2\right)$$

$$= -x^2\left(\frac{228}{x^4} + 2\right) + 4x^2 + x^2\left(\frac{228}{x^4} - 2\right) = 0\checkmark.$$

Also,

$$y(2) = -2\sqrt{\frac{228}{16} - 2} = -2\sqrt{\frac{228}{16} - \frac{32}{16}} = -2\sqrt{\frac{196}{16}} = -2\frac{14}{4} = -7.\checkmark$$

(4) **Solve the Initial Value Problem:**

$$y'' - 4y' + 4y = 0, \qquad y(0) = 1, \qquad \textbf{and} \qquad y'(0) = 4.$$

Put your answer in the form $y = y(x)$. Please check your answer.

We consider the characteristic equation $r^2 - 4r + 4 = 0$, which is $(r-2)^2 = 0$. The general solution of the Differential Equation is $y = c_1 e^{2x} + c_2 x e^{2x}$. We compute

$$y' = 2c_1 e^{2x} + c_2(2xe^{2x} + e^{2x})$$

$$1 = y(0) = c_1 \quad \text{and} \quad 4 = y'(0) = 2c_1 + c_2.$$

Thus, $c_1 = 1$ and $c_2 = 2$.

$$\boxed{y = e^{2x} + 2xe^{2x}.}$$

Check: Observe that

$$y' = 2e^{2x} + 4xe^{2x} + 2e^{2x} = 4e^{2x} + 4xe^{2x},$$

$$y'' = 8e^{2x} + 8xe^{2x} + 4e^{2x} = 12e^{2x} + 8xe^{2x},$$

and

$$\begin{aligned} &y'' - 4y' + 4y \\ &= (12e^{2x} + 8xe^{2x}) - 4(4e^{2x} + 4xe^{2x}) + 4(e^{2x} + 2xe^{2x}) \\ &= (12 - 16 + 4)e^{2x} + (8 - 16 + 4)xe^{2x} = 0; \checkmark \end{aligned}$$

$y(0) = 1\checkmark$ and $y'(0) = 4\ \checkmark$.

(5) **Suppose that a motorboat is moving at 50 feet per second when its motor suddenly quits and that 10 seconds later the boat has slowed to 30 feet per second. Assume that the only force acting on the boat is resistance and that resistance is proportional to velocity. How far will the boat coast in all?**

Let $v(t)$ be the velocity of the boat at time t. We take $t = 0$ to be the time when the motor failed. Eventually we also need to consider the position of the boat at time t, which we call $x(t)$. Observe that $x(t)$ is an anti-derivative of $v(t)$. We call the position of the boat when the motor failed to be $x(0) = 0$.

We are told that $\frac{dv}{dt} = -kv$ for some positive constant k; $v(0) = 50$, and $v(10) = 30$. We separate the variables and integrate

$$\frac{dv}{v} = -kdt$$

$$\ln|v| = -kt + C.$$

Exponentiate to learn that

$$|v| = e^C e^{-kt}$$

$$v = \pm e^C e^{-kt}$$

Let $K = \pm e^C$.

$$v = Ke^{-kt}$$

Plug in $t = 0$ to learn that $K = 50$

$$v = 50e^{-kt}.$$

We find k by plugging in $t = 10$

$$30 = 50e^{-10k}$$

$$\ln \frac{3}{5} = -10k$$

$$\frac{\ln \frac{3}{5}}{-10} = k.$$

Of course k also is equal to

$$k = \frac{\ln \frac{5}{3}}{10},$$

which is a positive number, as expected. Now that we know $v = \frac{dx}{dt}$; we integrate to find

$$x = \int 50e^{-kt} dt = \frac{50}{-k} e^{-kt} + C_2.$$

Plug in $t = 0$ to learn

$$0 = x(0) = \frac{50}{-k} + C_2$$

Thus

$$\frac{50}{k} = C_2.$$

The position of the boat at time t is

$$x(t) = \frac{50}{k}(1 - e^{-kt}).$$

The velocity never becomes zero; so the total distance traveled by the boat after the motor fails is

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{50}{k}(1 - e^{-kt}) = \frac{50}{k} = 50\frac{10}{\ln \frac{5}{3}}.$$

$$\boxed{\text{The boat coasts for } \frac{500}{\ln \frac{5}{3}} \text{ feet.}}$$