Math 242, Exam 2, Solution, Spring 2013

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

The solutions will be posted later today.

1. (10 points) **Find the general solution of** $y''' + 3y'' + 3y' + y = 0$**. Check your answer.**

This is a linear homogeneous DE with constant coefficients. We try $y = e^{rx}$. We must solve $r^3 + 3r^2 + 3r + 1 = 0$. This characteristic polynomial factors as $(r+1)^3 = 0$. The only root is $r = -1$, with multiplicity 3. The corresponding solutions of the DE are $y_1 = e^{-x}$, $y_2 = xe^{-x}$, and $y_3 = x^2e^{-x}$. The general solution of the DE is

$$\boxed{y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x}},$$

where c_1, c_2, and c_3 are arbitrary constants.

Check. We compute

$$\begin{aligned}
&y = e^{-x}(c_1 + c_2x + c_3x^2)\\
&y' = e^{-x}(c_2 + 2c_3x) - e^{-x}(c_1 + c_2x + c_3x^2) = e^{-x}(c_2 - c_1 + (2c_3 - c_2)x - c_3x^2)\\
&y'' = e^{-x}(2c_3 - c_2 - 2c_3x) - e^{-x}(c_2 - c_1 + (2c_3 - c_2)x - c_3x^2)\\
&\quad = e^{-x}(2c_3 - 2c_2 + c_1 + (-4c_3 + c_2)x + c_3x^2)\\
&y''' = e^{-x}((-4c_3 + c_2) + 2c_3x) - e^{-x}(2c_3 - 2c_2 + c_1 + (-4c_3 + c_2)x + c_3x^2)\\
&\quad = e^{-x}((-6c_3 + 3c_2 - c_1) + (6c_3 - c_2)x - c_3x^2).
\end{aligned}$$

Thus,

$$y''' + 3y'' + 3y' + y = \begin{cases} e^{-x}((-6c_3 + 3c_2 - c_1) + (6c_3 - c_2)x - c_3x^2)\\ +3e^{-x}(2c_3 - 2c_2 + c_1 + (-4c_3 + c_2)x + c_3x^2)\\ +3e^{-x}(c_2 - c_1 + (2c_3 - c_2)x - c_3x^2)\\ +e^{-x}(c_1 + c_2x + c_3x^2)\end{cases}$$

and this is zero.

2. (10 points) **Find the general solution of** $y'' - 4y' + 29y = 0$**. Check your answer.**

This is a linear homogeneous DE with constant coefficients. We try $y = e^{rx}$. We must solve $r^2 - 4r + 29 = 0$. Use the quadratic formula to see that $r = \frac{4 \pm \sqrt{16 - 4 \cdot 29}}{2} = \frac{4 \pm \sqrt{4(4-29)}}{2} = \frac{4 \pm 2\sqrt{-25}}{2} = 2 \pm 5i$. The corresponding solutions of the DE are $y_1 = e^{2x}\cos(5x)$ and $y_2 = e^{2x}\sin(5x)$. The general solution of the DE is

$$\boxed{y = c_1 e^{2x}\cos(5x) + c_2 e^{2x}\sin(5x)}$$

where c_1 and c_2 are arbitrary constants.

Check. We compute

$$\begin{aligned}
y &= e^{2x}(c_1\cos(5x) + c_2\sin(5x)) \\
y' &= e^{2x}(-5c_1\sin(5x) + 5c_2\cos(5x)) + 2e^{2x}(c_1\cos(5x) + c_2\sin(5x)) \\
&= e^{2x}((-5c_1 + 2c_2)\sin(5x) + (5c_2 + 2c_1)\cos(5x)) \\
y'' &= e^{2x}(5(-5c_1 + 2c_2)\cos(5x) - 5(5c_2 + 2c_1)\sin(5x)) \\
&\quad + 2e^{2x}((-5c_1 + 2c_2)\sin(5x) + (5c_2 + 2c_1)\cos(5x)) \\
&\quad = e^{2x}((-21c_1 + 20c_2)\cos(5x) + (-21c_2 - 20c_1)\sin(5x)).
\end{aligned}$$

Thus,

$$y'' - 4y' + 29y = \begin{cases} e^{2x}((-21c_1 + 20c_2)\cos(5x) + (-21c_2 - 20c_1)\sin(5x)) \\ -4e^{2x}((-5c_1 + 2c_2)\sin(5x) + (5c_2 + 2c_1)\cos(5x)) \\ +29e^{2x}(c_1\cos(5x) + c_2\sin(5x)) \end{cases}$$

and this is zero.

3. (10 points) **Solve the initial value problem** $y'' - y = 0$**,** $y(0) = 4$**, and** $y'(0) = 2$**. Check your answer.**

This is a linear homogeneous DE with constant coefficients. We try $y = e^{rx}$. We must solve $r^2 - 1 = 0$; so $(r-1)(r+1) = 0$. In other words, $r = 1$ or $r = -1$. The corresponding solutions of the DE are $y_1 = e^x$ and $y_2 = e^{-x}$. The general solution of the DE is

$$y = c_1 e^x + c_2 e^{-x},$$

where c_1 and c_2 are arbitrary constants. We use the initial conditions to evaluate the constants. We compute

$$y' = c_1 e^x - c_2 e^{-x}.$$

Plug in to learn

$$4 = y(0) = c_1 + c_2 \quad \text{and} \quad 2 = y'(0) = c_1 - c_2.$$

Add these two equations to learn that $6 = 2c_1$ or $3 = c_1$. The first equation minus the second equation gives $2 = 2c_2$ or $c_2 = 1$. The solution is $\boxed{y = 3e^x + e^{-x}}$.

Check. We compute

$$\begin{aligned} y &= 3e^x + e^{-x} \\ y' &= 3e^x - e^{-x} \\ y'' &= 3e^x + e^{-x} \end{aligned}$$

Thus $y'' - y = 3e^x + e^{-x} - (3e^x + e^{-x}) = 0$, $y(0) = 3+1 = 4$, and $y'(0) = 3-1 = 2$. ✓

4. (10 points) **Solve** $\frac{dy}{dx} = y^2 + y - 6$**,** $y(0) = y_0$**. Sketch your solution when** $0 \le x$ **for various values of** y_0**. Check your answer.**

We see that $\frac{dy}{dx}$ depends only on y and not on the independent variable x. It makes sense to look for equilibrium solutions of this differential equation and to determine if the equilibrium solutions are stable or unstable. At any rate $\frac{dy}{dx} = (y-2)(y+3)$. So $y = 2$ and $y = -3$ are equilibrium solutions of the DE. We draw a quick picture (figure 1) to see that $y = 2$ is an unstable equilibrium and $y = -3$ is a stable equilibrium. We are now able to "sketch our solution when $0 \le x$ for various values of y_0"; see figure 2. (This is pretty cool. We can draw the solutions before we found the solutions.)

Okay, so now we find the solutions. We solve

$$\frac{dy}{(y-2)(y+3)} = dx.$$

We solve

$$\frac{1}{5}\int\left(\frac{1}{y-2} - \frac{1}{y+3}\right)dy = \int dx.$$

Integrate to obtain:

$$\tfrac{1}{5}(\ln|y-2| - \ln|y+3|) = x + C$$

$$\ln\left|\tfrac{y-2}{y+3}\right| = 5x + 5C.$$

Exponentiate:

$$\frac{y-2}{y+3} = \pm e^{5C}e^{5x}.$$

Let $K = \pm e^{5C}$.

$$\frac{y-2}{y+3} = Ke^{5x}.$$

Plug in $x = 0$ to learn that $K = \frac{y_0 - 2}{y_0 + 3}$.

$$y - 2 = Ke^{5x}(y+3)$$

$$y(1 - Ke^{5x}) = 3Ke^{5x} + 2$$

$$y = \frac{3Ke^{5x} + 2}{1 - Ke^{5x}}$$

$$y = \frac{3\frac{y_0-2}{y_0+3}e^{5x} + 2}{1 - \frac{y_0-2}{y_0+3}e^{5x}}$$

$$y = \frac{3(y_0 - 2)e^{5x} + 2(y_0 + 3)}{(y_0 + 3) - (y_0 - 2)e^{5x}}$$

$$y = \frac{-3(2 - y_0) + 2(y_0 + 3)e^{-5x}}{(2 - y_0) + (y_0 + 3)e^{-5x}}$$

Notice that:

- If $y_0 = 2$ then $y(x) = 2$ for all x. (Of course, we already know that).
- Similarly, if $y_0 = -3$, then $y(x) = -3$ for all x. (Again, we already knew that.)
- If $y_0 < 2$, then the denominator never becomes 0. In this case, if one takes $\lim_{x \to \infty} y(x)$, then the limit is -3. (Once again, we knew that.)
- If $2 < y_0$, then the denominator is positive (indeed 5) when $x = 0$; but for large x the denominator is negative. Thus for some positive value for x, the denominator becomes zero. (The numerator is always positive in this case.) So for some positive x there is a vertical asymptote in the graph of $y = y(x)$. Once again, we drew that in our quick sketch.

5. (10 points) **The acceleration of a car is proportional to the difference between 250 ft/sec and the velocity of the car. If this car can accelerate from 0 to 100 ft/sec in 10 seconds, how long will it take for the car to accelerate from rest to 150 ft/sec?**

Let $v(t)$ be the velocity of the car (measured in ft/sec) at time t seconds. We are told that $\frac{dv}{dt} = k(250 - v)$. The initial condition is $v(0) = 0$. We are told that $v(10) = 100$. (This allows us to find k.) We are asked to find the time with $v(t) = 150$. We integrate $\int \frac{dv}{250-v} = \int k dt$ to see that

(*) $$-\ln(250 - v) = kt + C$$

The initial condition $v(0) = 0$ tells us that $-\ln 250 = C$. We plug in $v(10) = 100$ into (*) to see that $-\ln(250 - 100) = 10k - \ln 250$. It follows that

$$\ln 250 - \ln(150) = 10k$$

$$\ln \frac{250}{150} = 10k;$$

so, $\frac{\ln\frac{5}{3}}{10} = k$. We now find the time when $v(t) = 150$. Again, we use (*). We solve $-\ln(250 - 150) = kt + C$. We solve $-\ln(100) = (\frac{\ln\frac{5}{3}}{10})t - \ln 250$. We see that $t = \frac{\ln 250 - \ln 100}{\frac{\ln\frac{5}{3}}{10}} = 10\frac{\ln\frac{250}{100}}{\ln\frac{5}{3}} = \boxed{10\frac{\ln\frac{5}{2}}{\ln\frac{5}{3}}\text{sec}}$.