Math 242, Exam 2 Solution, Fall 2012

You should KEEP this piece of paper. Write everything on the blank paper provided. If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

Nothing may be on your desk except things that came from me. In particular, **no Calculators or Cell phones** may be on your desk.

Your work must be coherent and correct. **I expect you to solve initial value problems. Unexplained, random formulas will not be accepted!**

The solutions will be posted later today.

1. (8 points) **Suppose that a body moves through a resisting medium with resistance proportional to its velocity $v(t)$, so that $\frac{dv}{dt} = -kv$, for some positive constant k. Let $x(t)$ be the position of the object at time t. Let $v(0) = v_0$ and $x(0) = x_0$.**
   (a) **Find the velocity of the object at time t.**
   (a) **Find the position of the object at time t.**
   (c) **Find $\lim_{t\to\infty} x(t)$.**

(a) Separate the variables and integrate to learn $\int \frac{dv}{v} = \int -k\,dt$. It follows that $\ln|v| = -kt + C$. Exponentiate to see that $v = Ke^{-kt}$. Plug in $t = 0$ to obtain $v_0 = K$. Thus, $\boxed{v(t) = v_0 e^{-kt}}$

(b) Integrate again to learn $x(t) = \frac{v_0}{-k}e^{-kt} + C_1$. Plug in $t = 0$ to see that $x_0 = \frac{v_0}{-k} + C_1$; so $\boxed{x(t) = \frac{v_0}{-k}e^{-kt} + \frac{v_0}{k} + x_0.}$

(c) $\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(\frac{v_0}{-k}e^{-kt} + \frac{v_0}{k} + x_0) = \boxed{\frac{v_0}{k} + x_0}$

2. (7 points) **Consider the initial value problem $\frac{dy}{dx} = 2x + y^3$, $y(1) = 2$. Use Euler's method to approximate $y(3/2)$. Use two steps, each of size $1/4$.**

The picture is drawn elsewhere. The curve represents the correct solution of the initial value problem. We approximate the real solution with the two line segments. The line segment from $(1, 2)$ to (x_1, y_1) has slope equal to $(2x + y^3)|_{(1,2)}$. The line segment from (x_1, y_1) to (x_2, y_2) has slope equal to $(2x + y^3)|_{(x_1,y_1)}$. Of course, $(x_0, y_0) = (1, 2)$, $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, and y_2 is our approximation of $y(\frac{3}{2})$.

We see that $(2x+y^3)|_{(1,2)} = 10$; so, $\frac{y_1-y_0}{x_1-x_0} = 10$; so, $y_1 = y_0 + \frac{10}{4} = 2 + \frac{5}{2} = \frac{9}{2}$. We see that $(2x+y^3)|_{(\frac{5}{4},\frac{9}{2})} = \frac{5}{2} + (\frac{9}{2})^3$; so, $\frac{y_2-y_1}{x_2-x_1} = \frac{5}{2} + (\frac{9}{2})^3$; so, $y_2 = y_1 + (\frac{5}{2} + (\frac{9}{2})^3)\frac{1}{4} = \boxed{\frac{9}{2} + (\frac{5}{2} + (\frac{9}{2})^3)\frac{1}{4}}$.

3. (7 points) **Consider the Initial Value Problem $\frac{dx}{dt} = 9 - x^2$, $x(0) = x_0$.**
   (a) **Sketch the solution of this IVP for various values of x_0.**
   (b) **Solve the Initial Value Problem. Write your answer in the form $x = x(t)$.**

Is your answer to (b) consistent with your answer to (a)?

We do (b) here. Separate the variables and integrate:

$$\int \frac{dx}{(3-x)(3+x)} = \int dt$$

$$\frac{1}{6}\int \left[\frac{1}{3+x} + \frac{1}{3-x}\right] = \int dt$$

$$\frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| = t + C$$

$$\ln\left|\frac{3+x}{3-x}\right| = 6t + 6C$$

$$\frac{3+x}{3-x} = Ke^{6t}$$

$$3 + x = Ke^{6t}(3-x).$$

Plug $t = 0$ into this equation to learn $\frac{3+x_0}{3-x_0} = K$. We solve for x.

$$x(1 + Ke^{6t}) = 3Ke^{6t} - 3$$

$$x = \frac{3Ke^{6t} - 3}{1 + Ke^{6t}}$$

$$x = \frac{3\frac{3+x_0}{3-x_0}e^{6t} - 3}{1 + \frac{3+x_0}{3-x_0}e^{6t}}$$

$$\boxed{x = \frac{3(3+x_0)e^{6t} - 3(3-x_0)}{(3-x_0) + (3+x_0)e^{6t}}}.$$

4. (7 points) **Solve** $(x+y)\frac{dy}{dx} = 1$ **. (Your solution may be given as an implicit function. (In other words, you are not required to write your solution in the form** $y = y(x)$ **.) Check your answer.**

We make the linear substitution $v = x + y$. Observe that $\frac{dv}{dx} = 1 + \frac{dy}{dx}$. The DE becomes

$$v(-1 + \frac{dv}{dx}) = 1.$$

Separate the variables:

$$\frac{dv}{dx} = \frac{1}{v} + 1$$

$$\frac{dv}{dx} = \frac{1+v}{v}$$

$$\int \frac{v}{1+v} dv = \int dx$$

$$\int \frac{(v+1)-1}{1+v} dv = \int dx$$

$$\int 1 - \frac{1}{1+v} dv = \int dx$$

$$v - \ln|1+v| = x + C$$

$$x + y - \ln|1+x+y| = x + C$$

$$y - \ln|1+x+y| = C.$$

Any function $y(x)$ which satisfies $y - \ln|1+x+y| = C$ is a solution of the DE.

Check Use implicit differentiation to compute the derivative of our solution. That is, take $\frac{d}{dx}$ of both sides of $y - \ln|1+x+y| = C$ to see that

$$\frac{dy}{dx} - \frac{1 + \frac{dy}{dx}}{1+x+y} = 0$$

Now solve for $\frac{dy}{dx}$. Multiply both sides by $1 + x + y$ to get

$$(1+x+y)\frac{dy}{dx} - (1 + \frac{dy}{dx}) = 0$$

$$(x+y)\frac{dy}{dx} = 1,$$

as desired.

5. (7 points) **Solve** $x(x+y)\frac{dy}{dx}+y(3x+y)=0$**. Write your solution in the form** $y=y(x)$**. Check your answer.**

Each term in this equation has degree two in x and y. We make a homogeneous substitution $v=\frac{y}{x}$. So $xv=y$ and $x\frac{dv}{dx}+v=\frac{dy}{dx}$. Divide the original equation by x^2 to obtain

$$(1+\tfrac{y}{x})\tfrac{dy}{dx}+\tfrac{y}{x}(3+\tfrac{y}{x})=0.$$

Make the substitution:

$$(1+v)(x\tfrac{dv}{dx}+v)+v(3+v)=0.$$

Separate the variables:

$$x\tfrac{dv}{dx}=\tfrac{-4v-2v^2}{1+v}$$

$$(\bigstar)\qquad \int\tfrac{1+v}{2v+v^2}dv=-2\int\tfrac{dx}{x}.$$

We apply the technique of partial fractions. We look for numbers A and B with

$$\frac{1+v}{2v+v^2}=\frac{A}{v}+\frac{B}{v+2}.$$

Multiply both sides by $2v+v^2=v(v+2)$ to obtain:

$$1+v=A(v+2)+Bv.$$

Plug in $B=0$ to learn that $A=\frac{1}{2}$. Plug in $v=-2$ to learn that $B=\frac{1}{2}$. We check this:

$$\frac{1}{2}\left[\frac{1}{v}+\frac{1}{v+2}\right]=\frac{1}{2}\left[\frac{v+2+v}{v(v+2)}\right]=\frac{1}{2}\left[\frac{2v+2}{v(v+2)}\right]=\frac{v+1}{v^2+2v}.$$

We see that $(\bigstar)$ is

$$\int\tfrac{1}{2}\left[\tfrac{1}{v}+\tfrac{1}{v+2}\right]dv=-2\int\tfrac{dx}{x}.$$

$$\int\left[\tfrac{1}{v}+\tfrac{1}{v+2}\right]dv=-4\int\tfrac{dx}{x}.$$

$$\ln|v|+\ln|v+2|=-4\ln|x|+C.$$

Exponentiate to get $v(v+2)=Kx^{-4}$. So $v^2+2v-Kx^{-4}=0$. Use the quadratic formula: if $av^2+bv+c=0$, then

$$v=\frac{-b\pm\sqrt{b^2-4ac}}{2}a=\frac{-2\pm\sqrt{4+4Kx^{-4}}}{2}=-1\pm\sqrt{1+Kx^{-4}}$$

So, $y = xv = x(-1 \pm \sqrt{1 + Kx^{-4}})$. Thus, $\boxed{y = -x \pm \sqrt{x^2 + Kx^{-2}}}$.

Check We check $y = -x + \sqrt{x^2 + Kx^{-2}}$. We compute

$$x(x+y)\frac{dy}{dx} + y(3x+y)$$

$$= x(x - x + \sqrt{x^2 + Kx^{-2}})(-1 + \tfrac{2x - 2Kx^{-3}}{2\sqrt{x^2+Kx^{-2}}}) + (-x + \sqrt{x^2 + Kx^{-2}})(3x - x + \sqrt{x^2 + Kx^{-2}})$$

$$= x(\sqrt{x^2 + Kx^{-2}})(-1 + \tfrac{x - Kx^{-3}}{\sqrt{x^2+Kx^{-2}}}) + (-x + \sqrt{x^2 + Kx^{-2}})(2x + \sqrt{x^2 + Kx^{-2}})$$

$$= -x\sqrt{x^2 + Kx^{-2}} + x^2 - Kx^{-2} - 2x^2 + x\sqrt{x^2 + Kx^{-2}} + x^2 + Kx^{-2} = 0,$$

as desired.

6. (7 points) **A pitcher of buttermilk initially at 40° C is to be cooled by setting it on the front porch, where the temperature is 15° C. Suppose that the temperature of the buttermilk has dropped to 30° C after 25 minutes. When will the temperature of the buttermilk reach 20° C? (Recall that Newton's Law of Cooling states that the rate at which an object cools is proportional to the difference between the temperature of the object and the temperature of the surrounding medium.)**

Let $T(t)$ be the temperature of the buttermilk at time t. We take $t = 0$ to be the moment that the buttermilk was put on the porch. We measure time in minutes and temperature in degrees C. We are told that $\frac{dT}{dt} = -k(T - 15)$ for some positive constant k. We are also told that $T(0) = 40$ and $T(25) = 30$. We want t_{end} with $T(t_{\text{end}}) = 20$. We separate the variables and integrate to learn $\ln|T - 15| = -kt + C$. Exponentiate to obtain $T - 15 = Ke^{-kt}$, for $K = \pm e^C$. Plug in $T(0) = 40$ to learn that $K = 25$. Thus, $T - 15 = 25e^{-kt}$ for all time t. We plug in $T(25) = 30$ to see that $30 - 15 = 25e^{-25k}$; hence $\frac{3}{5} = e^{-25k}$ and $\ln\frac{3}{5} = -25k$. So $\frac{\ln\frac{3}{5}}{-25} = k$ Now we can find t_{end} because $20 = T(t_{\text{end}}) = 15 + 25e^{-kt_{\text{end}}}$. Thus, $5 = 25e^{-kt_{\text{end}}}$ and $\frac{1}{5} = e^{-kt_{\text{end}}}$. Take the logarithm of both sides to obtain

$$\ln(\tfrac{1}{5}) = -kt_{\text{end}}.$$

Thus,

$$t_{\text{end}} = -\ln(\tfrac{1}{5})\tfrac{1}{k} = -\ln\tfrac{1}{5}\tfrac{-25}{\ln(\frac{3}{5})} = \ln 5\tfrac{25}{\ln(\frac{5}{3})} = \boxed{\frac{25\ln(5)}{\ln 5 - \ln 3}\text{min}}.$$

7. (7 points) **Suppose the velocity of a motorboat coasting in water satisfies the differential equation $\frac{dv}{dt} = kv^2$. The initial speed of the motorboat is $v(0) = 10$ meters per second (m/s), and v is decreasing at the rate 1 m/s^2 when $v = 5$ m/s. How long does it take for the velocity of the boat to decrease to 1 m/s?**

We separate the variables and integrate to see that $\int \frac{dv}{v^2} = \int k dt$; so, $-1/v = kt + C$. Plug in $t = 0$ to learn that $-1/10 = C$. Let t_1 be the time when $v = 5$. We are told that at time t_1, we have $\frac{dv}{dt}(t_1) = -1$. Plug t_1 into $\frac{dv}{dt} = kv^2$ to learn:

$$-1 = \frac{dv}{dt}(t_1) = kv(t_1)^2 = k(25).$$

So, $k = -1/25$. Thus,

$$v = \frac{-1}{kt + C} = \frac{-1}{\frac{-1}{25}t + \frac{-1}{10}}.$$

Multiply top and bottom by -50 to get

$$v = \frac{50}{2t + 5}.$$

We find the time t_2, when $v(t_2) = 1$:

$$1 = \frac{50}{2t_2 + 5}$$

$$2t_2 + 5 = 50$$

$$2t_2 = 45$$

$$\boxed{t_2 = 22.5 \text{seconds}}$$