Math 242, Exam 2, Spring, 2018

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted on Saturday. The exam will be returned in class on Tuesday.

No Calculators or Cell phones.

(1) **A motor boat is moving at 40 feet per second when its motor suddenly quits and 10 seconds later the boat has slowed to 20 feet/second. The only force acting on the boat is resistance and resistance is proportional to velocity. How far will the boat coast in all?**

Let $x(t)$ equal the position of the boat at time t, where t measures the amount of time since the motor quit. We are told

$$x'' = -kx', \quad x'(0) = 40, \quad x'(10) = 20, \quad \text{and} \quad x(0) = 0,$$

for some positive constant k. If you like, let $v = x'$. Separate the variables in $\frac{dv}{dt} = -kv$ and integrate $\int \frac{dv}{v} = \int -kdt$:

$$\begin{aligned} \ln|v| &= -kt + C \\ |v| &= e^{-kt+C} \\ v &= Ke^{-kt} \\ x' &= Ke^{-kt} \end{aligned}$$

Plug in $t = 0$ to learn $K = 40$. So,

$$x' = 40e^{-kt}$$

Plug in $t = 10$ to learn

$$\begin{aligned} 20 &= 40e^{-10k} \\ \frac{1}{2} &= e^{-10k} \\ -\ln 2 &= -10k \\ \frac{\ln 2}{10} &= k \end{aligned}$$

Integrate with respect to t to see that

$$x = \frac{40}{-k}e^{-kt} + C_1$$

Plug in $t = 0$ to see that
$$0 = \frac{40}{-k} + C_1$$
so $C_1 = \frac{40}{k}$ and $x(t) = \frac{40}{k} - \frac{40}{-k}e^{-kt}$. We see that x' is never zero; but $\lim\limits_{t\to\infty} x' = 0$. The total distance traveled by the boat is
$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{40}{k} - \frac{40}{-k}e^{-kt} = \frac{40}{k} = \frac{40}{\frac{\ln 2}{10}} = \boxed{\frac{400}{\ln 2}\text{feet}}.$$

(2) **Solve $yy' + x = \sqrt{x^2+y^2}$. Express your answer in the form $y(x)$. Check your answer.**

This is a homogeneous problem. Divide both sides by x to write the problem as
$$\frac{y}{x}y' + 1 = \sqrt{1 + \left(\frac{y}{x}\right)^2}.$$
Let $v = \frac{y}{x}$. In other words, $xv = y$. Take the derivative with respect to x to see that $xv' + v = y'$. We must solve
$$v(xv' + v) + 1 = \sqrt{1+v^2}.$$
We must solve
$$xv\frac{dv}{dx} = \sqrt{1+v^2} - v^2 - 1.$$
We must solve
$$v\frac{dv}{\sqrt{1+v^2} - v^2 - 1} = \frac{dx}{x}.$$
Integrate both sides. Let $w = 1 + v^2$. It follows that $dw = 2vdv$. We must solve
$$\frac{1}{2}\int \frac{dw}{\sqrt{w} - w} = \ln|x| + C.$$
We have
$$\ln|x| + C = \frac{1}{2}\int \frac{dw}{\sqrt{w}(1-\sqrt{w})}.$$
Let $u = \sqrt{w}$. We have $du = \frac{1}{2}w^{-1/2}dw$.
We have
$$\ln|x| + C = \int \frac{du}{1-u} = -\ln|1-u| = -\ln|1-\sqrt{w}| = -\ln|1-\sqrt{1+v^2}|$$
$$= -\ln\left|1 - \sqrt{1+\left(\frac{y}{x}\right)^2}\right| = -\ln\left|\frac{x-\sqrt{x^2+y^2}}{x}\right| = -\ln\left|x - \sqrt{x^2+y^2}\right| + \ln|x|.$$
Subtract $\ln|x|$ from both sides:
$$C = -\ln\left|x - \sqrt{x^2+y^2}\right|$$

or

$$\ln\left|x-\sqrt{x^2+y^2}\right| = -C.$$

Exponentiate. Let K be the new constant e^{-C}. We have

$$x-\sqrt{x^2+y^2} = K;$$

so $x-K=\sqrt{x^2+y^2}$ and $(x-K)^2=x^2+y^2$ and $\boxed{\pm\sqrt{(x-K)^2-x^2}=y.}$

Check: We check $y=+\sqrt{(x-K)^2-x^2}$, with $K\le x$. We see that

$$y' = \frac{2(x-K)-2x}{2\sqrt{(x-K)^2-x^2}} = \frac{-K}{\sqrt{(x-K)^2-x^2}}.$$

So, $yy'+x=-K+x$. On the other hand,

$$\sqrt{x^2+y^2}=\sqrt{x^2+(x-K)^2-x^2}=\sqrt{(x-K)^2}=x-K.$$

Thus, $yy'+x=\sqrt{y^2+x^2}$ as required. ✓

(3) **Consider the initial value problem $\frac{dy}{dx}=x+\frac{2}{y}$, $y(1)=3$. Use Euler's method to approximate $y(12/10)$. Use two steps, each of size $1/10$. Label all intermediate calculations clearly and correctly.**

Let $F(x,y)=x+\frac{2}{y}$, $x_0=1$, $x_1=11/10$, $x_2=12/10$, and $y_0=3$. We find y_1 and y_2 so that

$$\frac{y_1-y_0}{x_1-x_0}=F(x_0,y_0)$$

and

$$\frac{y_2-y_1}{x_2-x_1}=F(x_1,y_1)$$

Then y_2 is our approximation of $y(12/10)$. We have

$$y_1=3+(1/10)(1+2/3)=3+1/6=19/6$$

and

$$y_2=y_1+(1/10)(x_1+\frac{2}{y_1})=19/6+(1/10)(11/10+12/19).$$

$\boxed{\text{Our approximation of } y(12/10) \text{ is } y_2=19/6+(1/10)(11/10+12/19).}$

(4) **Solve the Initial Value Problems $\frac{dx}{dt}=x(x-4)$ with $x(0)=x_0$. Draw some of the solutions.**

We separate the variables and integrate:

$$(\dagger)\qquad \int\frac{dx}{x(x-4)}=\int dt$$

Use the technique of partial fractions to see that

$$\frac{1}{x(x-4)}=\frac{A}{x}+\frac{B}{x-4}.$$

$$1 = A(x-4) + Bx.$$

Plug in $x = 4$ to see that $1/4 = B$. Plug in $x = 0$ to see that $A = -1/4$. Check that

$$\tfrac{1}{4}[\tfrac{1}{(x-4)} - \tfrac{1}{x}] = \tfrac{1}{4}\tfrac{x-(x-4)}{x(x-4)} = \tfrac{1}{x(x-4)},$$

as desired. We write (†) as

$$\frac{1}{4}\int \frac{1}{(x-4)} - \frac{1}{x} = \int dt$$

$$\frac{1}{4}(\ln|x-4| - \ln|x|) = t + C$$

$$\ln\left|\frac{x-4}{x}\right| = 4t + 4C$$

$$\left|\frac{x-4}{x}\right| = e^{4t}e^{4C}$$

$$\frac{x-4}{x} = Ke^{4t}$$

where $K = \pm e^{4C}$. Plug in $t = 0$ to see that

$$\frac{x_0 - 4}{x_0} = K.$$

At any rate,

$$x - 4 = Ke^{4t}x$$

$$x(1 - Ke^{4t}) = 4$$

$$x = \frac{4}{(1 - Ke^{4t})}$$

$$x = \frac{4}{(1 - \frac{x_0-4}{x_0}e^{4t})}$$

$$\boxed{x(t) = \frac{4x_0}{x_0 - (x_0 - 4)e^{4t}}.}$$

So, $x(t) = 4$, for all t, is a solution of the DE and $x(t) = 0$ for all t is a solution of the DE.

With repect to graphing the solutions, it is best to look at the differential equation: $\frac{dx}{dt} = x(x-4)$.

If $4 < x_0$, then $\frac{dx}{dt}$ is always positive and the solution increases away from $x = 4$. If $0 \leq x_0 \leq 4$, then the $\frac{dx}{dt}$ is negative and the solution decreases toward $x = 0$. If x_0 is negative then $\frac{dx}{dt}$ is positive and the solution increases toward $x = 0$. The picture is on the picture page.

Check Take $\frac{d}{dt}$ of $x = 4x_0(x_0 - (x_0 - 4)e^{4t})^{-1}$ to obtain

$$\frac{dx}{dt} = 4x_0(-1)(x_0 - (x_0 - 4)e^{4t})^{-2}(-4(x_0 - 4)e^{4t})$$

$$= \frac{4x_0(-1)(-4(x_0-4)e^{4t})}{(x_0-(x_0-4)e^{4t})^2}$$
$$= \frac{16x_0(x_0-4)e^{4t}}{(x_0-(x_0-4)e^{4t})^2}$$

On the other hand,

$$x(x-4) = 4x_0(x_0-(x_0-4)e^{4t})^{-1}(4x_0(x_0-(x_0-4)e^{4t})^{-1}-4)$$
$$= 4x_0(x_0-(x_0-4)e^{4t})^{-1}(4x_0(x_0-(x_0-4)e^{4t})^{-1}-4(x_0-(x_0-4)e^{4t})(x_0-(x_0-4)e^{4t})^{-1})$$
$$= 4x_0(x_0-(x_0-4)e^{4t})^{-2}(4x_0-4(x_0-(x_0-4)e^{4t}))$$
$$= \frac{4x_0(4x_0-4(x_0-(x_0-4)e^{4t}))}{(x_0-(x_0-4)e^{4t})^2}$$

(5) **Solve the Initial Value Problem**

$$y''+y=3x, \quad y(0)=2, \quad y'(0)=-2.$$

You might want to notice that $y_1=\sin x$ **and** $y_2=\cos x$ **are solutions of the corresponding homogeneous problem and** $y_{\textbf{particular}}=3x$ **is a solution of the given Differential Equation. Check your answer.**

The general solution of the differential equation is $y=c_1\sin x+c_2\cos x+3x$. We compute $y'=c_1\cos x-c_2\sin x+3$. Plug in the initial condition to learn

$$2=c_1(0)+c_2(1)+3(0);$$

thus, $c_2=2$; and

$$-2=c_1(1)+c_2(0)+3;$$

thus $c_1=-5$. The solution is

$$\boxed{y=-5\sin x+2\cos x+3x}$$

<u>Check</u>

$$y=-5\sin x+2\cos x+3x$$
$$y'=-5\cos x-2\sin x+3$$
$$y''=5\sin x-2\cos x$$

Thus

$$y+y''=(-5\sin x+2\cos x+3x)+(5\sin x-2\cos x)=3x\checkmark$$

$$y(0)=-5(0)+2(1)+3(0)=2\checkmark$$
$$y'(0)=-5(1)-2(0)+3=-2\checkmark$$