Math 242, Exam 2, Spring, 2017 11:40 class solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today. The exams will be returned in class on Tuesday, Feb. 28.

No Calculators or Cell phones.

(1) **Consider the initial value problem $\frac{dy}{dx} = y + \frac{1}{x}$, $y(2) = 1$. Use Euler's method to approximate $y(22/10)$. Use two steps, each of size $1/10$.**

Let $f(x,y) = y + \frac{1}{x}$, $(x_0, y_0) = (2, 1)$, $x_1 = \frac{21}{10}$, and $x_2 = \frac{22}{10}$. Define y_1 so that the slope of the line joining (x_0, y_0) to (x_1, y_1) is $f(x_0, y_0)$. Define y_2 so that the slope of the line joining (x_1, y_1) to (x_2, y_2) is $f(x_1, y_1)$. Then y_2 is our approximation of $y(\frac{22}{10})$.

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$$

$$\frac{y_1 - 1}{\frac{1}{10}} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$y_1 = 1 + \left(\frac{1}{10}\right)\left(\frac{3}{2}\right) = 1 + \frac{3}{20} = \frac{23}{20}.$$

$$\frac{y_2 - y_1}{x_2 - x_1} = f(x_1, y_1)$$

$$\frac{y_2 - \frac{23}{20}}{\frac{1}{10}} = \left(\frac{23}{20}\right) + \frac{10}{21}.$$

$$y_2 = \frac{23}{20} + \frac{1}{10}\left(\left(\frac{23}{20}\right) + \frac{10}{21}\right).$$

$$\boxed{\text{Our approximation of } y(\tfrac{22}{10}) \text{ is } y_2 = \tfrac{23}{20} + \tfrac{1}{10}\left(\left(\tfrac{23}{20}\right) + \tfrac{10}{21}\right).}$$

(2) **Consider a tank with 200 liters of salt-water solution, 30 grams of which is salt. A brine solution with a concentration of 1 gram of salt per liter is pouring into the tank at the rate of 4 liters/minute. The "well-mixed" solution pours out of the tank at a rate of 5 liters/minute. Find the amount of salt in the tank at time t. SET UP THE INITIAL VALUE PROBLEM. DO NOT SOLVE THE INITIAL VALUE PROBLEM.**

Let $x(t)$ equal the number of grams of salt in the tank at time t minutes. We are told $x(0) = 30$. The rate of change of x is the rate in minus the rate out. The rate in is

$$\frac{1 \text{ gram}}{\text{liter}} \frac{4 \text{ liters}}{\text{min}}.$$

The rate out is

$$\frac{x(t) \text{ grams}}{200 - t \text{ liter}} \frac{5 \text{ liters}}{\text{min}}.$$

The Initial Value Problem is

$$\boxed{\frac{dx}{dt} = 4 - \frac{5x}{200 - t}, \quad x(0) = 30.}$$

(3) **Solve $\frac{dy}{dx} - (4x - y + 1)^2 = 0$. Express your answer in the form $y = y(x)$. Please check your answer.**

We make a linear substitution. Let $v = 4x - y + 1$. Compute $\frac{dv}{dx} = 4 - \frac{dy}{dx}$. The Differential Equation becomes

$$4 - \frac{dv}{dx} - v^2 = 0$$

$$4 - v^2 = \frac{dv}{dx}$$

$$-dx = \frac{dv}{v^2 - 4}$$

We find constants A and B with

$$\frac{1}{(v-2)(v+2)} = \frac{A}{v-2} + \frac{B}{v+2}$$

Multiply both sides by $(v - 2)(v + 2)$ to obtain

$$1 = A(v + 2) + B(v - 2).$$

Plug in $v = 2$ to learn that $A = \frac{1}{4}$. Plug in $v = -2$ to learn that $B = -\frac{1}{4}$. We verify that

$$\frac{1}{(v-2)(v+2)} = \frac{1}{4}\left[\frac{1}{v-2} - \frac{1}{v+2}\right]$$

The right side is

$$\frac{1}{4}\left[\frac{v+2-(v-2)}{(v-2)(v+2)}\right] = \frac{1}{4}\left[\frac{4}{(v-2)(v+2)}\right] = \frac{1}{(v-2)(v+2)}. \checkmark$$

We integrate both sides of

$$-\int dx = \frac{1}{4}\int \left[\frac{1}{v-2} - \frac{1}{v+2}\right] dv$$

$$-x + C = \frac{1}{4}(\ln|v-2| - \ln|v+2|)$$

$$-4x + 4C = \ln\left|\frac{v-2}{v+2}\right|$$

$$e^{4C}e^{-4x} = \left|\frac{v-2}{v+2}\right|$$

$$\pm e^{4C}e^{-4x} = \frac{v-2}{v+2}$$

Let $K = \pm e^{4C}$.

$$Ke^{-4x} = \frac{v-2}{v+2}$$

$$(v+2)Ke^{-4x} = v-2$$

$$2Ke^{-4x} + 2 = v(1 - Ke^{-4x})$$

$$\frac{2Ke^{-4x}+2}{(1-Ke^{-4x})} = v$$

$$\frac{2Ke^{-4x}+2}{(1-Ke^{-4x})} = 4x - y + 1$$

$$\boxed{y = 4x + 1 - \frac{2Ke^{-4x}+2}{(1-Ke^{-4x})}.}$$

Check. We plug the proposed answer into the left side of the differential equation to get

$$\frac{dy}{dx} - (4x - y + 1)^2$$

$$= 4 - \left[(2Ke^{-4x}+2)(-1)\frac{+4Ke^{-4x}}{(1-Ke^{-4x})^2} + \frac{-8Ke^{-4x}}{(1-Ke^{-4x})}\right] - \left(\frac{2Ke^{-4x}+2}{(1-Ke^{-4x})}\right)^2$$

$$= 4 - \frac{\left[-(2Ke^{-4x}+2)4Ke^{-4x} - 8Ke^{-4x}(1-Ke^{-4x})\right] + \left(2Ke^{-4x}+2\right)^2}{(1-Ke^{-4x})^2}$$

$$= 4 - \frac{\left[-8K^2e^{-8x} - 8Ke^{-4x} - 8Ke^{-4x} + 8K^2e^{-8x}\right] + \left(4K^2e^{-8x} + 8Ke^{-4x} + 4\right)}{(1-Ke^{-4x})^2}$$

$$= 4 - \frac{\left[-16Ke^{-4x}\right] + \left(4K^2e^{-8x} + 8Ke^{-4x} + 4\right)}{(1-Ke^{-4x})^2}$$

$$= 4 - \frac{\left(4K^2e^{-8x} - 8Ke^{-4x} + 4\right)}{(1-Ke^{-4x})^2}$$

$$= 4 - \frac{4\left(K^2e^{-8x} - 2Ke^{-4x} + 1\right)}{(1-Ke^{-4x})^2}$$

$$= 4 - \frac{4(Ke^{-2x}-1)^2}{(1-Ke^{-4x})^2} = 4 - 4 = 0. \checkmark$$

(4) **Solve $\frac{dy}{dx} = (y-2)(3-y)$. Express your answer in the form $y = y(x)$. Draw some of the solutions of this Differential Equation for various values of $y(0)$.**

We separate the variables and integrate:

$$\int \frac{dy}{(y-2)(3-y)} = \int dx.$$

We look for constants A and B with

$$\frac{A}{y-2} + \frac{B}{3-y} = \frac{1}{(y-2)(3-y)}.$$

Multiply both sides by $(y-2)(3-y)$ to obtain

$$A(3-y) + B(y-2) = 1.$$

Plug in $y = 3$ to learn that $B = 1$. Plug in $y = 2$ to learn that $A = 1$. We verify that

$$\frac{1}{y-2} + \frac{1}{3-y} = \frac{1}{(y-2)(3-y)}.$$

The left side is

$$\frac{3-y+y-2}{(y-2)(3-y)} = \frac{1}{(y-2)(3-y)}.\checkmark$$

We integrate both sides of

$$\int \left[\frac{1}{y-2} + \frac{1}{3-y}\right] dy = \int dx$$

to obtain

$$\ln|y-2| - \ln|3-y| = x + C$$
$$\ln\left|\frac{y-2}{3-y}\right| = x + C$$
$$\left|\frac{y-2}{3-y}\right| = e^c e^x$$
$$\frac{y-2}{3-y} = \pm e^c e^x$$

Let K represent the constant $K = \pm e^c$. This is a good time to see that

$$\frac{y(0)-2}{3-y(0)} = K.$$

At any rate

$$y - 2 = Ke^x(3-y)$$
$$y(1+Ke^x) = 3Ke^x + 2$$
$$\boxed{y = \frac{3Ke^x+2}{1+Ke^x}, \text{ with } K = \frac{y(0)-2}{3-y(0)}}$$

The picture is drawn elsewhere. We see that $y(x) = 2$ and $y(x) = 3$ are equilibrium solutions of the DE. Other solutions are attracted to $y = 3$ and flee $y = 2$.

(5) **Find all constants r for which $y = e^{rx}$ a solution of $y'' - 4y' + 3y = 0$. Find a constant A with $y = Ae^{2x}$ a solution of $y'' - 4y' + 3y = e^{2x}$. What is the general solution of $y'' - 4y' + 3y = e^{2x}$?**

We plug e^{rx} into $y'' - 4y' + 3y = 0$ to obtain

$$r^2 e^{rx} - 4re^{rx} + 3e^{rx} = 0.$$

So,

$$e^{rx}(r^2 - 4r + 3) = 0;$$

but e^{rx} is never zero; so

$$(r^2 - 4r + 3) = 0$$

or

$$(r-3)(r-1) = 0.$$

We have shown that if $y = e^{rx}$ is a solution of $y'' - 4y' + 3y = 0$, then $r = 1$ or $r = 3$ and $y = e^x$ or $y = e^{3x}$.

If $y = Ae^{2x}$ a solution of $y'' - 4y' + 3y = e^{2x}$, then

$$4Ae^{2x} - 8Ae^{2x} + 3Ae^{2x} = e^{2x};$$

hence

$$-Ae^{2x} = e^{2x}$$

and $A = -1$.

We have found two linearly independent solutions of the homogeneous problem and one particular solution of the non-homogeneous problem. It follows that the general solution of $y'' - 4y' + 3y = e^{2x}$ is

$$\boxed{y = c_1 e^x + c_2 e^{3x} - e^{2x}.}$$