Math 242, Exam 1, Summer 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems. Each problem is worth 10 points. SHOW your work. $\boxed{\textit{CIRCLE}}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Solve** $2x\frac{dy}{dx} = y + x + 1$. **Express your answer in the form** $y(x)$. **Check your answer.**

This is a first order linear DE:

$$(\bigstar) \qquad \frac{dy}{dx} - \frac{1}{2x}y = \frac{x+1}{2x}$$

of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

with $P(x) = -\frac{1}{2x}$ and $Q(x) = \frac{x+1}{2x}$. The magic multiplier is

$$\mu(x) = e^{\int P(x)dx} = e^{\int -\frac{1}{2x}dx} = e^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}.$$

Multiply both sides of ($\bigstar$) by $\frac{1}{\sqrt{x}}$ to get

$$\frac{1}{\sqrt{x}}\frac{dy}{dx} - \frac{1}{2x^{3/2}}y = \frac{x+1}{2x^{3/2}}.$$

We are thrilled to see that the left side of the most recent line is $\frac{d}{dx}(\frac{1}{\sqrt{x}}y)$ (because that is the WHOLE POINT of calculating μ.) So, now we have

$$\frac{d}{dx}(\frac{1}{\sqrt{x}}y) = \frac{1}{2}(x^{-1/2} + x^{-3/2}).$$

Integrate both sides with respect to x:

$$(\frac{1}{\sqrt{x}}y) = \frac{1}{2}(2x^{1/2} - 2x^{-1/2}) + C.$$

Multiply both sides by $\sqrt{x}$ to get:

$$\boxed{y = x - 1 + C\sqrt{x}.}$$

We plug our answer into the original DE and see that

$$2x\frac{dy}{dx} = 2x(1 + \frac{C}{2\sqrt{x}}) = 2x + C\sqrt{x}.$$

On the other hand,

$$y + x + 1 = x - 1 + C\sqrt{x} + x + 1 = 2x + C\sqrt{x}.$$

These are equal. Our answer is correct.

2. **Solve** $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}$ **,** $y(5) = 2$ **. Express your answer in the form** $y(x)$ **. Check your answer.**

Separate the variables and integrate

$$\int 2y dy = \int \frac{x}{\sqrt{x^2-16}} dx$$

$$y^2 = \sqrt{x^2-16} + C.$$

Plug in the initial condition:

$$4 = \sqrt{25-16} + C$$

$$4 = 3 + C$$

$$1 = C$$

Our solution is $\boxed{y = \sqrt{\sqrt{x^2-16}+1}}$. We check this. $y(5) = \sqrt{\sqrt{25-16}+1} = \sqrt{3+1} = 2$ ✓. Now we take the derivative

$$\frac{dy}{dx} = \frac{\frac{2x}{2\sqrt{x^2-16}}}{2\sqrt{\sqrt{x^2-16}+1}} = \frac{\frac{x}{\sqrt{x^2-16}}}{2\sqrt{\sqrt{x^2-16}+1}} = \frac{x}{2\sqrt{x^2-16}\sqrt{\sqrt{x^2-16}+1}}.$$

We see that

$$2y\frac{dy}{dx} = 2\sqrt{\sqrt{x^2-16}+1}\frac{x}{2\sqrt{x^2-16}\sqrt{\sqrt{x^2-16}+1}}$$

$$= \frac{x}{\sqrt{x^2-16}},$$

as desired.

3. **A tank initially contains 100 gallons of brine in which 50 pounds of salt are dissolved. A brine containing 2 pounds per gallon of salt runs into the tank at the rate of 5 gallons per minute. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gallons per minute. How many pounds of salt are in the tank at time** t **?**

Let $x(t)$ be the number of pounds of salt in the tank at time t. We are told that $x(0) = 50\text{lbs}$. We know that salt is entering the tank at the rate of $2\frac{\text{lbs}}{\text{gal}} \times \frac{5\text{gal}}{\text{min}} =$

$10\frac{\text{gal}}{\text{min}}$ and salt is leaving the tank at the rate of $\frac{x}{100+t}\frac{\text{lbs}}{\text{gal}}\frac{4\text{gal}}{\text{min}} = \frac{4x}{100+t}\frac{\text{lbs}}{\text{min}}$. We must solve the initial value problem:

$$\frac{dx}{dt} = 10 - \frac{4x}{100+t}, \quad x(0) = 50.$$

This is a First Order Linear Differential Equation:

$$(*) \qquad \frac{dx}{dt} + \frac{4}{100+t}x = 10;$$

of the form

$$\frac{dx}{dt} + P(t)x = Q(t)$$

with $P(t) = \frac{4}{100+t}$ and $Q(t) = 10$. The magic multiplier is

$$\mu(t) = e^{\int P(t)dt} = e^{\int \frac{4}{100+t}dt} = e^{\int 4\ln(100+t)} = (100+t)^4.$$

We multiply both sides of (*) by $(100+t)^4$ to obtain

$$(100+t)^4\frac{dx}{dt} + 4(100+t)^3x = 10(100+t)^4.$$

We are thrilled to see that the left side of the most recent line is $\frac{d}{dt}((100+t)^4x)$ (because that is the WHOLE POINT of calculating μ.) So, now we have

$$\frac{d}{dt}((100+t)^4x) = 10(100+t)^4.$$

Integrate both sides with respect to t:

$$((100+t)^4x) = 2(100+t)^5 + C.$$

Let us plug in the initial condition at this point: $100^4 \cdot 50 = 2(100)^5 + C$; so, $C = 50 \cdot 100^4 - 2(100^5) = 100^4(50 - 200) = -150(100)^4$ and

$$((100+t)^4x) = 2(100+t)^5 - 150(100)^4.$$

Divide both sides by $(100+t)^4$ to learn

$$\boxed{x(t) = 2(100+t) - \frac{150(100)^4}{(100+t)^4}.}$$

We check that $x(0) = 2(100) - \frac{150(100)^4}{(100)^4} = 200 - 150 = 50$ as desired. Also,

$$\begin{aligned}\frac{dx}{dt} + \frac{4}{100+t}x &= 2 - \frac{150(100)^4}{(100+t)^5}(-4) + \frac{4}{100+t}\left(2(100+t) - \frac{150(100)^4}{(100+t)^4}\right)\\ &= 2 + 8 = 10,\end{aligned}$$

as desired.

4. **Consider the Initial Value problem**

$$\text{(IVP)} \qquad \frac{dy}{dx} = \frac{-x+\sqrt{x^2+4y}}{2}, \quad y(2) = -1.$$

(a) **Is $y_1(x) = 1-x$ a solution of (IVP)? Show your work. Your answer must make sense.**

We see that $y_1(2) = 1-2 = -1$. We compute $y_1'(x) = -1$. Plug y_1 into the right side of (IVP) to obtain:

$$\frac{-x+\sqrt{x^2+4y_1}}{2} = \frac{-x+\sqrt{x^2+4(1-x)}}{2} = \frac{-x+\sqrt{x^2-4x+4}}{2}$$

$$= \frac{-x+\sqrt{(x-2)^2}}{2}$$

We know that

$$\sqrt{(x-2)^2} = \begin{cases} x-2 & \text{if } 2 \le x \\ -(x-2) & \text{if } x < 2. \end{cases}$$

We consider $2 \le x$; so we have

$$\frac{-x+\sqrt{x^2+4y_1}}{2} = \frac{-x+(x-2)}{2} = \frac{-2}{2} = -1 = y_1'(x);$$

so, yes, $y_1(x)$ is a solution of (IVP) for $2 \le x$.

(b) **Is $y_2(x) = \frac{-x^2}{4}$ a solution of (IVP)? Show your work. Your answer must make sense.**

We see that $y_2(2) = \frac{-(2)^2}{4} = -1$. We compute $y_2'(x) = \frac{-x}{2}$. Plug y_2 into the right side of (IVP) to obtain:

$$\frac{-x+\sqrt{x^2+4y_2}}{2} = \frac{-x+\sqrt{x^2+4\left(\frac{-x^2}{4}\right)}}{2} = \frac{-x+\sqrt{x^2-x^2}}{2} = \frac{-x}{2} = y_2'(x);$$

so, yes, $y_2(x)$ is a solution of (IVP) for all x.

(c) **State the Existence and Uniqueness Theorem for first order differential equations.**

Consider the Initial Value Problem IVP: $y' = F(x,y)$ with $y(x_0) = y_0$.

(i) If F is continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a solution on some interval containing x_0.

(ii) If F and F_y are both continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a unique solution on some interval containing x_0.

(d) **Comment on your answer to (c) in light of your answers to (a) and (b).**

The "F" for us, in this problem, is $\frac{-x+\sqrt{x^2+4y}}{2}$. This F is continuous provided $0 \le x^2 + 4y$. Our initial condition $(x_0, y_0) = (2, -1)$ is right on the edge of the region where F is continuous because $2^2 + 4(-1) = 0$. So it is not possible to draw a rectangle around our initial condition (with our initial condition in the interior) and stay within the region where F is continuous. Thus, the Existence and Uniqueness Theorem does not promise anything about our problem! (If you are keeping strict score $F_y = \frac{1}{\sqrt{x^2+4y}}$. So, F_y does not even exist at our initial condition. So the second part of the Existence and Uniqueness Theorem also does not promise anything about our problem.) It is interesting to notice that our initial condition is the left-most point for which y_1 works. One prefers to have solutions which makes sense of both sides of the initial point.

5. **Newton's law of cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium. Thus, if an object is taken from an oven at $350°$ F and left to cool in a room at $65°$ F, then its temperature T after t hours will satisfy the differential equation**

$$\frac{dT}{dt} = k(T - 65).$$

If the temperature fell to $250°$ F after one hour, what will it be after 3 hours?

We know $T(0) = 350°$ and $T(1) = 250°$. We want $T(3)$. Separate the variables in the differential equation:

$$\int \frac{dT}{T - 65} = \int k\,dt.$$

Integrate both sides to get

$$\ln|T - 65| = kt + C.$$

Exponentiate to see $|T-65| = e^C e^{kt}$ or $T-65 = \pm e^C e^{kt}$. Let $\mathcal{C}$ be the constant $\pm e^C$. So $T-65 = \mathcal{C}e^{kt}$. Plug in $t=0$ to see that $350-65=\mathcal{C}$. So, $T-65 = 285e^{kt}$. Plug in $t=1$ to see that $250-65 = 285e^k$. It follows that $\frac{185}{285} = e^k$ and $\ln\frac{185}{285} = k$. Thus,

$$T(t) = 65 + 285e^{t\ln\frac{185}{285}}.$$

We conclude that

$$\boxed{T(3) = 65 + 285e^{3\ln\frac{185}{285}}}.$$

The answer is given in degrees F.