Math 242, Exam 1, Spring 2012

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. There are **5** problems **ON TWO SIDES**. Each problem is worth 10 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

1. **Solve** $x^3 + 3y - x\frac{dy}{dx} = 0$ **. Express your answer in the form** $y(x)$ **. Check your answer.** This is the First Order Linear DE

$$\frac{dy}{dx} + \frac{-3}{x}y = x^2.$$

(The DE has the form $y' + P(x)y = Q(x)$.) We multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{-3}{x}dx} = e^{-3\ln x} = \frac{1}{x^3}.$$

The DE becomes

$$\frac{1}{x^3}\frac{dy}{dx} + \frac{-3}{x^4}y = \frac{1}{x}.$$

In other words, the DE is

$$\frac{d\left(\frac{1}{x^3}y\right)}{dx} = \frac{1}{x}.$$

Multiply both sides by dx and integrate to learn

$$\frac{1}{x^3}y = \ln|x| + C.$$

The solution is

$$\boxed{y = x^3\ln|x| + Cx^3}$$

Check. We plug $y = x^3\ln x + Cx^3$ back into the DE to see if it works. We compute

$$\frac{dy}{dx} = x^3\frac{1}{x} + 3x^2\ln x + 3Cx^2.$$

It follows that

$$x^3 + 3y - x\frac{dy}{dx} = x^3 + 3(x^3\ln x + Cx^3) - x(x^3\frac{1}{x} + 3x^2\ln x + 3Cx^2),$$

and this does indeed equal zero.

2. **Solve** $2xy\frac{dy}{dx} = 4x^2+3y^2$ **. Express your answer in the form** $y(x)$ **. Check your answer.**

This is a homogeneous problem. (Every term has degree 2 in the symbols x and y .) Divide both sides of the equation by x^2 to get $2\frac{y}{x}\frac{dy}{dx} = 4+3(\frac{y}{x})^2$. Let $v = \frac{y}{x}$. View the DE as a DE with v a function of x . Notice that $xv = y$; so, $xv'+v = y'$ (where $'$ means $\frac{d}{dx}$). The DE is $2v(xv'+v) = 4+3v^2$. Subtract $2v^2$ from both sides to obtain $2vxv' = 4+v^2$. Divide both sides by $(4+v^2)x$ and multiply both sides by dx to get

$$\frac{2v\,dv}{4+v^2} = \frac{dx}{x}.$$

Now integrate to obtain $\ln(4+v^2) = \ln|x| + C$. Exponentiate and let K be the constant $\pm e^C$ to obtain $4+v^2 = Kx$. Thus, $v^2 = Kx-4$, $v = \pm\sqrt{Kx-4}$, and $\boxed{y = \pm x\sqrt{Kx-4}}$.

Check. We plug $y = x\sqrt{Kx-4}$ back into the DE to see if it works. We have $\frac{dy}{dx} = x\frac{K}{2\sqrt{Kx-4}} + \sqrt{Kx-4}$. It follows that

$$2xy\frac{dy}{dx} = 2xx\sqrt{Kx-4}\left[x\frac{K}{2\sqrt{Kx-4}} + \sqrt{Kx-4}\right] = Kx^3 + 2x^2(Kx-4) = 3Kx^3 - 8x^2.$$

On the other hand,

$$4x^2+3y^2 = 4x^2 + 3x^2(Kx-4) = 3Kx^2 - 8x^2.$$

Thus, $y = x\sqrt{Kx-4}$ really is a solution of the DE.

3. **A 200-gallon tank is full of a solution containing 25 pounds of salt. Starting at time** $t=0$ **, pure water is added to the tank at a rate of 10 gallons per minute, and the well-stirred solution is withdrawn at the same rate. Find the number of pounds of salt in the tank at time** t **.**

Let $x(t)$ be the number of pounds of salt in the tank at time t . We are told that $x(0) = 25$. We know that salt is entering the tank at the rate of 0 and salt is leaving the tank at the rate of $\frac{x}{200}\frac{\text{lbs}}{\text{gal}}\frac{(-10)\text{gal}}{\text{min}} = \frac{-x}{20}\frac{\text{lbs}}{\text{min}}$. We must solve the initial value problem:

$$\frac{dx}{dt} = \frac{-x}{20}, \quad x(0) = 25.$$

We can separate the variables: $\frac{dx}{x} = \frac{-1}{20}dt$. Integrate both sides to get $\ln x = \frac{-t}{20} + C$. Exponentiate and let $K = e^C$ to obtain $x = Ke^{\frac{-t}{20}}$. Plug in $t=0$ to learn $25 = K$. We conclude that $\boxed{x(t) = 25e^{\frac{-t}{20}}}$.

4.

(a) **State the Existence and Uniqueness Theorem for first order differential equations.**

Consider the Initial Value Problem IVP: $y' = f(x,y)$ with $y(x_0) = y_0$.

(i) If f is continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a solution on some interval containing x_0.

(ii) If f and f_y are both continuous on some rectangle that contains (x_0, y_0) in its interior, then IVP has a unique solution on some interval containing x_0.

(b) **What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem**

$$(1+x^2)y' = (1+y)^2 \quad y(0) = 0?$$

This DE has the form $y' = f(x,y)$ with $f(x,y) = \frac{(1+y)^2}{(1+x^2)}$. We see that f and $f_y = \frac{2(1+y)}{(1+x^2)}$ are both continuous everywhere. We conclude the given initial problem has a unique solution on some interval containing $x = 0$.

(c) **Solve the Initial Value Problem of part (b).**

Separate the variables: $\int \frac{dy}{(1+y)^2} = \int \frac{dx}{x^2+1}$. Integrate to get

$$-(1+y)^{-1} = \arctan x + C.$$

Plug in $y(0) = 0$ to see that $C = -1$. So the solution is

$$\boxed{\frac{-1}{\arctan x - 1} - 1 = y}.$$

Check: We see that $y(0) = 1 - 1 = 0$. ✓ We also see that

$$y' = \frac{1}{(1-\arctan x)^2} \frac{1}{1+x^2};$$

so,

$$(1+x^2)y' = \frac{1}{(1-\arctan x)^2}.$$

On the other hand,

$$(y+1)^2 = \left(\frac{1}{1 - 1\arctan x}\right)^2.$$

These agree. ✓

(d) **What does the Existence and Uniqueness Theorem tell you about the Initial Value Problem**

$$(1+x^2)y' = (1+y)^2 \quad y(0) = -1?$$

This DE has the form $y' = f(x,y)$ with $f(x,y) = \frac{(1+y)^2}{(1+x^2)}$. We see that f and $f_y = \frac{2(1+y)}{(1+x^2)}$ are both continuous everywhere. We conclude the given initial problem has a unique solution on some interval containing $x = 0$.

(e) **Solve the Initial Value Problem of part (d).**

The technique we used in (c) does not work because we can not divide by $y+1$ if y is sometimes equal to -1. On the other hand, we are guaranteed that a unique solution exists on some interval near $x = 0$. We have $(0,-1)$ on our solution and as we leave this point we leave with slope 0 because $y' = \frac{(1+y)^2}{(1+x^2)}$ so $y'(0,-1) = 0$. Travel a little bit along the line $y = -1$. Ask the DE which way you should go. That is, plug (near $0,-1$) into y'. Again, $y' = 0$. In fact the unique solution to this IVP is $\boxed{y = -1}$. This function satisfies the initial condition and y' is always zero so $(1+x^2)0 = (0)^2$ does indeed hold!

5. **Newton's law of cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium. Thus, if an object is taken from an oven at $400°$ F and left to cool in a room at $70°$ F, then its temperature T after t hours will satisfy the differential equation**

$$\frac{dT}{dt} = k(T - 70).$$

If the temperature fell to $200°$ F after one hour, what will it be after 4 hours? (You may leave "ln" in your answer.)

We know $T(0) = 400°$ and $T(1) = 200°$. We want $T(4)$. Separate the variables in the differential equation:

$$\int \frac{dT}{T-70} = \int k\,dt.$$

Integrate both sides to get

$$\ln|T-70| = kt + C.$$

Exponentiate to see $|T-70| = e^C e^{kt}$ or $T-70 = \pm e^C e^{kt}$. Let $\mathcal{C}$ be the constant $\pm e^C$. So $T-70 = \mathcal{C}e^{kt}$. Plug in $t=0$ to see that $400-70 = \mathcal{C}$. So, $T-70 = 330e^{kt}$. Plug in $t=1$ to see that $200-70 = 330e^k$. It follows that $\frac{130}{330} = e^k$ and $\ln\frac{13}{33} = k$. Thus,

$$T(t) = 70 + 330e^{t\ln\frac{13}{33}}.$$

We conclude that

$$\boxed{T(4) = 70 + 330e^{4\ln\frac{13}{33}}}.$$

The answer is given in degrees F.