Math 242, Exam 1, Solution, Spring 2013

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: turn the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. SHOW your work. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible.

No Calculators or Cell phones.

The solutions will be posted later today.

1. **(8 points) Find all constants r for which $y = e^{rx}$ is a solution of $2y'' - 5y' + 3y = 0$. Your work must be coherent and meaningful.**

Compute $y' = re^{rx}$ and $y'' = r^2e^{rx}$. So $y = e^{rx}$ is a solution of the DE provided

$$2r^2e^{rx} - 5re^{rx} + 3e^{rx} = 0.$$

That is,

$$e^{rx}(2r^2 - 5r + 3) = 0.$$

The function e^{rx} is always positive and is never 0. So $y = e^{rx}$ is a solution of the differential equation if and only if $2r^2 - 5r + 3 = 0$. We can factor this quadratic: $(2r - 3)(r - 1) = 0$. We conclude that $\boxed{r = 1 \text{ or } r = \frac{3}{2}}$.

2. **(7 points) On the planet Gzyx, a ball dropped from a height of 80 ft hits the ground in 5 seconds. If a ball is dropped from the top of a 200-ft-tall building on Gzyx, how long will it take to hit the ground? With what speed will it hit? I expect you to solve initial value problems. Unexplained, random formulas will not be accepted! (Recall that Newton's Law of Motion states that if $F(t)$ is the force acting on an object moving in a straight line at time t, m is the mass of the object, and $a(t)$ is the acceleration of the object at time t, then $F = ma$. The only force acting on this ball on planet Gzyx is the force of gravity and this force is constant.) Your work must be coherent and meaningful.**

Let $x(t)$ be the height of the ball above the ground at time t. Measure t in seconds and x in feet. We are told that $x''(t) = -k$ for some positive constant k. For the first event, we have $x(0) = 80$, $x'(0) = 0$, and $x(5) = 0$. For the second event, we have $x(0) = 200$ and $x'(0) = 0$. We want to find t_1 with $x(t_1) = 0$. We also want to find $x'(t_1)$.

We first think about the first event. Integrate to learn $x'(t) = -kt + C_1$. Plug in $x'(0) = 0$ to learn that $C_1 = 0$. Integrate again to learn $x(t) = -kt^2/2 + C_2$.

Plug in $x(0)=80$ to learn $C_2=80$. So, $x(t)=-kt^2/2+80$. Plug in $x(5)=0$ to learn $k=\frac{32}{5}$.

Now turn to the second event. Integrate twice and evaluate the constants to learn that $x'(t)=-kt$ and $x(t)=-kt^2/2+200$; with $k=\frac{32}{5}$; so, $x(t)=-\frac{16}{5}t^2+200$. Solve $0=x(t_1)=-\frac{16}{5}t_1^2+200$ to learn that $t_1=\frac{5}{2}\sqrt{10}$ and $x'(t_1)=-16\sqrt{10}$.

It takes the second ball $\frac{5}{2}\sqrt{10}$ seconds to hit the ground. The ball is traveling downward at the speed $16\sqrt{10}$ feet per second when it hits the ground.

3. **(7 points) When the brakes are applied to a certain car, the acceleration of the car is $-k\frac{\mathrm{m}}{\mathrm{s}^2}$ for some positive constant k. Suppose that the car is traveling at the velocity $v_0\frac{\mathrm{m}}{\mathrm{s}}$ when the brakes are first applied and that the brakes continue to be applied until the car stops. I expect you to solve initial value problems. Unexplained, random formulas will not be accepted! Your work must be coherent and meaningful.**

   (a) **Find the distance that the car travels between the moment that the brakes are first applied and the moment when the car stops. (Your answer will be expressed in terms of k and v_0.)**

   Let $x(t)$ be the position of the car at time t. We take $t=0$ to be the moment that the brakes are applied. So $v(0)=v_0$ and $x(0)=0$. We are told $x''=-k$. We integrate and plug in the points to see $v(t)=-kt+v_0$ and $x(t)=-kt^2/2+v_0t$. Let t_s be the time when the car stops. We have $0=v(t_s)=-kt_s+v_0$. Thus, $t_s=v_0/k$. The distance traveled while the brakes were applied is

$$x(t_s)=x(v_0/k)=-k(v_0/k)^2/2+v_0(v_0/k)=(v_0^2/k)(1-1/2)=\boxed{\frac{v_0^2}{2k}}.$$

   (b) **How does the stopping distance change if the initial velocity is changed to $6v_0$?**

   $\boxed{\text{The stopping distance is multiplied by } 6^2}$ if v_0 is replaced by $6v_0$.

4. **(7 points) Solve $\frac{dy}{dx}=(4x+y)^2$. Express your answer in the form $y=y(x)$. Check your answer. Your work must be coherent and meaningful.**

We make the linear substitution $v=4x+y$. It follows that $\frac{dv}{dx}=4+\frac{dy}{dx}$. The problem becomes

$$\frac{dv}{dx}-4=v^2$$

$$\frac{dv}{4+v^2} = dx$$

Integrate both sides:

$$\frac{1}{2}\arctan(\frac{v}{2}) = x + C$$

$$\arctan(\frac{v}{2}) = 2x + 2C$$

$$\frac{v}{2} = \tan(2x + 2C)$$

$$v = 2\tan(2x + 2C)$$

$$4x + y = 2\tan(2x + 2C)$$

$$y = 2\tan(2x + 2C) - 4x$$

Let K be the constant $2C$.

$$\boxed{y = 2\tan(2x + K) - 4x.}$$

Check. We see that

$$\frac{dy}{dx} = 4\sec^2(2x+K) - 4 = 4\tan^2(2x+K) = ([2\tan(2x+K) - 4x] + 4x)^2 = (y+4x)^2. \checkmark$$

5. **(7 points) Solve** $x^2\frac{dy}{dx} + 2xy = 5y^3$ **. Express your answer in the form** $y = y(x)$ **. Check your answer. Your work must be coherent and meaningful.**

This is a Bernoulli equation. Let $v =$ "y^{1-n}" $= y^{-2}$. It follows that $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$. Multiply the original equation by y^{-3} to obtain:

$$x^2\frac{dy}{dx}y^{-3} + 2xy^{-2} = 5.$$

Substitute to obtain

$$\frac{x^2}{-2}\frac{dv}{dx} + 2xv = 5.$$

Multiply both sides by $-2x^{-2}$ to obtain:

$$\frac{dv}{dx} - 4x^{-1}v = -10x^{-2}.$$

This is a first order linear problem. Multipliy both sides of the equation by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{-4}{x}dx} = e^{-4\ln x} = x^{-4}$$

to obtain

$$x^{-4}\frac{dv}{dx} - 4x^{-5}v = -10x^{-6}.$$

This is

$$\frac{d}{dx}(x^{-4}v) = -10x^{-6}.$$

Integrate both sides:

$$x^{-4}v = 2x^{-5} + C$$

$$v = \frac{2}{x} + Cx^4$$

$$y^{-2} = \frac{2}{x} + Cx^4$$

$$\frac{1}{\frac{2}{x} + Cx^4} = y^2$$

$$\boxed{\pm\sqrt{\frac{x}{2 + Cx^5}} = y}$$

Check: We check $\sqrt{\frac{x}{2+Cx^5}} = y$. Plug this proposed answer into the left side of the DE to get

$$x^2\frac{dy}{dx} + 2xy = x^2\frac{1}{2}\sqrt{\frac{2+Cx^5}{x}}\,\frac{(2+Cx^5) - x5Cx^4}{(2+cx^5)^2} + 2x\sqrt{\frac{x}{2+Cx^5}}$$

$$= \frac{x\sqrt{x}}{(2+Cx^5)^{\frac{3}{2}}}\left[\frac{1}{2}((2+Cx^5) - 5Cx^5) + 2(2+Cx^5)\right] = \frac{x\sqrt{x}}{(2+Cx^5)^{\frac{3}{2}}}[5] = 5y^3.\checkmark$$

6. **(7 points) Solve $y\frac{dy}{dx} + x = \sqrt{y^2+x^2}$. Express your answer in the form $y = y(x)$. Check your answer. Your work must be coherent and meaningful.**

This is a homogeneous problem. Divide both sides by x to write the problem as

$$\frac{y}{x}y' + 1 = \sqrt{1 + \left(\frac{y}{x}\right)^2}.$$

Let $v = \frac{y}{x}$. In other words, $xv = y$. Take the derivative with respect to x to see that $xv' + v = y'$. We must solve

$$v(xv' + v) + 1 = \sqrt{1 + v^2}.$$

We must solve

$$xv\frac{dv}{dx} = \sqrt{1+v^2} - v^2 - 1.$$

We must solve

$$v\frac{dv}{\sqrt{1+v^2} - v^2 - 1} = \frac{dx}{x}.$$

Integrate both sides. Let $w = 1 + v^2$. It follows that $dw = 2vdv$. We must solve

$$\frac{1}{2}\int \frac{dw}{\sqrt{w} - w} = \ln|x| + C.$$

We have

$$\ln|x| + C = \frac{1}{2}\int \frac{dw}{\sqrt{w}(1-\sqrt{w})}.$$

Let $u = \sqrt{w}$. We have $du = \frac{1}{2}w^{-1/2}dw$. We have

$$\ln|x| + C = \int \frac{du}{1-u} = -\ln|1-u| = -\ln|1-\sqrt{w}| = -\ln|1-\sqrt{1+v^2}|$$

$$= -\ln\left|1 - \sqrt{1+\left(\frac{y}{x}\right)^2}\right| = -\ln\left|\frac{x-\sqrt{x^2+y^2}}{x}\right| = -\ln\left|x-\sqrt{x^2+y^2}\right| + \ln|x|.$$

Subtract $\ln|x|$ from both sides:

$$C = -\ln\left|x-\sqrt{x^2+y^2}\right|$$

or

$$\ln\left|x-\sqrt{x^2+y^2}\right| = -C$$

Exponentiate. Let K be the new constant e^{-C}. We have

$$x - \sqrt{x^2+y^2} = K;$$

so $x - K = \sqrt{x^2+y^2}$ and $(x-K)^2 = x^2 + y^2$ and $\boxed{\pm\sqrt{(x-K)^2 - x^2} = y.}$

Check: We check $y = +\sqrt{(x-K)^2 - x^2}$, with $K \le x$. We see that

$$y' = \frac{2(x-K) - 2x}{2\sqrt{(x-K)^2 - x^2}} = \frac{-K}{\sqrt{(x-K)^2 - x^2}}.$$

So, $yy' + x = -K + x$. On the other hand,

$$\sqrt{x^2+y^2} = \sqrt{x^2 + (x-K)^2 - x^2} = \sqrt{(x-K)^2} = x - K.$$

Thus, $yy' + x = \sqrt{y^2+x^2}$ as required. ✓

7. **(7 points) Consider two tanks. The first tank has a volume of 100 gals. of brine. The second tank has a volume of 200 gals. of brine. Each tank initially contains 50 lbs. of salt. Pure water flows into the first tank at the rate of 5 gal./min. The well mixed solution flows out of tank 1 and into tank 2 at the rate of 5 gal./min. The well mixed solution flows out of tank 2 at the rate of 5 gal./min. Your work must be coherent and meaningful.**

(a) **How much salt is in the first tank at time t?**

Let $x(t)$ be the number of pounds of salt in tank 1 at time t. We know that $\frac{dx}{dt}$ is the rate in minus the rate out. We also know that the rate in is 0 and the rate out is $\frac{x\text{lbs}}{100\text{gal}} \times \frac{5\text{gal}}{\text{min}}$. The DE for x is $\frac{dx}{dt} = -x/20$. Separate the variables, integrate, evaluate the constant: $\boxed{x(t) = 50e^{-t/20}}$.

(b) **How much salt is in the second tank at time t?**

Let $y(t)$ be the number of pounds of salt in tank 2 at time t. We know that $\frac{dy}{dt}$ is the rate in minus the rate out. We also know that the rate in is $\frac{x\text{lbs}}{100\text{gal}} \times \frac{5\text{gal}}{\text{min}} = \frac{5e^{-t/20}}{2}$ and the rate out is $\frac{y\text{lbs}}{200\text{gal}} \times \frac{5\text{gal}}{\text{min}} = y/40$. The DE for y is $\frac{dy}{dt} = \frac{5e^{-t/20}}{2} - y/40$. Write the problem as $\frac{dy}{dt} + y/40 = \frac{5e^{-t/20}}{2}$. Multiply both sides of the equation by $\mu(t) = e^{\int 1/40dt} = e^{t/40}$ to obtain: $e^{t/40}\frac{dy}{dt} + e^{t/40}y/40 = e^{t/40}\frac{5e^{-t/20}}{2}$. This is

$$\frac{d}{dt}(e^{t/40}y) = \frac{5}{2}e^{-t/40}.$$

Integrate both sides with respect to t to obtain

$$e^{t/40}y = (-40)\frac{5}{2}e^{-t/40} + C.$$

Plug in $t = 0$ to learn $50 = -100 + C$; so $150 = C$ and

$$\boxed{y = -100e^{-t/20} + 150e^{-t/40}}.$$

Of course, we check that $y(0) = -100 + 150 = 50$ and that the proposed solution actually satisfies the DE. The left side becomes:

$$\frac{dy}{dt} = 5e^{-t/20} - \frac{15}{4}e^{-t/40}.$$

The right side becomes:

$$\begin{aligned}\frac{5e^{-t/20}}{2} - y/40 &= \frac{5e^{-t/20}}{2} - (-100e^{-t/20} + 150e^{-t/40})/40 \\ &= \frac{5e^{-t/20}}{2} + \frac{5e^{-t/20}}{2} - 150e^{-t/40}/40.\end{aligned}$$

The two sides are equal.