Math 242, Exam 1, Spring, 2021

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, send a picture of your solutions to

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The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **Solve the Initial Value Problem:**

$$\frac{dy}{dx} = \frac{x}{(x^2+1)^2} \qquad \textbf{and} \qquad y(0) = 1.$$

Please check your answer.

Integrate both sides with respect to x to learn that

$$y = \frac{-1}{2(x^2+1)} + C.$$

Plug in the initial condition to learn

$$1 = \frac{-1}{2} + C;$$

so, $\frac{3}{2} = C$. The answer is

$$\boxed{y = \frac{-1}{2(x^2+1)} + \frac{3}{2}.}$$

Check. Observe that

$$\frac{dy}{dx} = \frac{-1}{2}(-1)(x^2+1)^{-2}2x = \frac{x}{(x^2+1)^2}\checkmark$$

and when $x = 0$, then

$$y = \frac{-1}{2(1)} + \frac{3}{2} = 1\checkmark.$$

(2) **Solve the Initial Value Problem:**

$$\frac{dy}{dx} = \frac{2xy}{x^2+1} \qquad \textbf{and} \qquad y(0) = 3.$$

Please check your answer.

Separate the variables, then integrate each side:

$$\int \frac{dy}{y} = \frac{2x}{x^2+1}dx$$

or

$$\ln|y| = \ln(x^2+1) + C.$$

Exponentiate to obtain

$$e^{\ln|y|} = e^{\ln(x^2+1)+C}$$

or

$$|y| = e^C(x^2+1)$$

or

$$y = \pm e^C(x^2+1)$$

Plug in the initial condition to learn $3 = \pm e^C$. We conclude that

$$\boxed{y = 3(x^2+1)}.$$

Check. We see that $\frac{dy}{dx} = 6x$ and that

$$\frac{2xy}{x^2+1} = \frac{2x(3(x^2+1))}{x^2+1},$$

which is also equal to $6x$. ✓We also see that $y(0) = 3(0^2+1) = 3$. ✓

(3) **Consider the Initial Value Problem:**

$$\frac{dy}{dx} = \frac{x}{y} \qquad \textbf{and} \qquad y(1) = 2$$

Use Euler's method to approximate $y(\frac{3}{2})$. Use two steps. Make each step size be $1/4$.

Let $f(x,y) = \frac{x}{y}$, $x_0 = 1$, $y_0 = 2$, $x_1 = \frac{5}{4}$, and $x_2 = \frac{3}{2}$. We find y_1 and y_2 so that the slope of the line joining (x_0,y_0) to (x_1,y_1) is $f(x_0,y_0)$ and the slope of the line joining (x_1,y_1) to (x_2,y_2) is $f(x_1,y_1)$. Then y_2 is our approximation of $y(\frac{3}{2})$. We get to work. Observe that

$$\frac{1}{2} = \frac{x_0}{y_0} = f(x_0,y_0) = \frac{y_1-y_0}{x_1-x_0} = \frac{y_1-2}{\frac{1}{4}}.$$

It follows that

$$\frac{17}{8} = 2 + \frac{1}{8} = y_1.$$

Observe further that

$$\frac{10}{17} = \frac{\frac{5}{4}}{\frac{17}{8}} = \frac{x_1}{y_1} = f(x_1, y_1) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - \frac{17}{8}}{\frac{1}{4}}.$$

It follows that

$$\frac{17}{8} + \frac{5}{34} = \frac{17}{8} + \frac{1}{4}\frac{10}{17} = y_2.$$

We conclude that

$$\boxed{y\left(\frac{3}{2}\right) \text{ is approximately equal to } \frac{17}{8} + \frac{5}{34}.}$$

(4) **Newton's law of cooling states that the rate at which an object cools is proportional to the difference between the temperature of the object and the temperature of the surrounding medium. When a cake is removed from an oven, its temperature is measured at 300 degrees. Three minutes later its temperature is 200 degrees. How long will it take for the cake to cool off to a temperature of 80 degrees? The room temperature is 70 degrees.**

Let $T(t)$ be the temperature of the cake at time t. We consider $t = 0$ to be the moment the cake was taken from the oven. Measure t in minutes and T in degrees. Newton's Law of Cooling states that $\frac{dT}{dt} = k(T - 70)$, for some constant k. The initial condition is $T(0) = 300$. The information $T(3) = 200$ can be used to determine the constant k.

We first solve the Differential Equation $\frac{dT}{dt} = k(T - 70)$. We separate the variables and integrate both sides

$$\int \frac{dT}{T - 70} = \int k dt$$

$$\ln |T - 70| = kt + C$$

Exponentiate both sides to obtain

$$|T - 70| = e^C e^{kt}$$

$$T = 70 + \pm e^C e^{kt}$$

Plug in the initial condition to learn

$$300 = T(0) = 70 + \pm e^C.$$

Thus $\pm e^C$ is equal to 230 and

$$T = 70 + 230e^{kt}.$$

Plug in $T(3) = 200$ to learn

$$200 = 70 + 230e^{3k}.$$

Thus

$$130 = 230e^{3k}$$

and $\frac{1}{3}\ln\frac{13}{23} = k$. Now we can answer the question. The temperature of the cake is 80 when

$$80 = 70 + 230e^{kt}.$$

We solve for t:

$$\ln\frac{1}{23} = kt$$

or

$$\boxed{t = 3\frac{\ln\frac{1}{23}}{\ln\frac{13}{23}}\text{minutes.}}$$

(5) **A 1500 gallon tank initially contains 600 gallons of water with 5 pounds of salt dissolved in it. A salt water solution which contains 15 pounds of salt per gallon enters the tank at a rate of 9 gallons per hour. If a well mixed solution leaves the tank at a rate of 6 gallons per hour, how much salt is in the tank when the tank overflows?**

Let $X(t)$ be the number of pounds of salt in the tank at time t hours. We are told that $X(0) = 5$. We want $X(300)$. (The tank starts with 600 gallons of liquid. Every hour the tank gains 3 gallons of liquid. It will take 300 hours until the tank is full with $600 + 3(300) = 1500$ gallons of liquid.)

The differential equation about $X(t)$ is obtained from the fact that $\frac{dX}{dt}$ is equal to the rate at which salt enters the tank minus the rate at which salt leaves the tank. The problem tells us that the rate at which salt enters the tank is

$$9\frac{\text{gallons}}{\text{hour}} \cdot 15\frac{\text{pounds}}{\text{gallon}}$$

The problem also tells us that the rate at which the solution leaves the tank is 6 gallons per hour. We must figure out how many pounds of salt are in each gallon of solution in the tank at time t; but we can do this. The number of pounds of salt in the tank at time t is $X(t)$. The number of gallons of solution in the tank at time t hours is $600 + 3t$. Thus, the rate at which salt leaves the tank is

$$6\frac{\text{gallons}}{\text{hour}} \times \frac{X(t)}{(600+3t)}\frac{\text{pounds}}{\text{gallon}}.$$

We now have an Initial Value Problem:

$$\frac{dX}{dt} = 135 - \frac{6X(t)}{(600+3t)} \quad \text{and} \quad X(0) = 5.$$

The problem is the First Order Linear problem

$$\frac{dX}{dt} + \frac{6X(t)}{(600+3t)} = 135. \tag{1}$$

We think of this problem as

$$\frac{dX}{dt} + P(t)X = Q(t),$$

where $P(t) = \frac{6}{600+3t}$ and $Q(t) = 135$. We multiply both sides of (1) by[1]

$$\mu(t) = e^{\int P(t)dt} = e^{2\ln(600+3t)} = (600+3t)^2$$

to obtain

$$(600+3t)^2\frac{dX}{dt} + 6(600+3t)X = 135(600+3t)^2. \tag{2}$$

Observe that the left side of (2) is the derivative, with respect to t, of $(600+3t)^2X$. Integrate both sides of (2) with respect to t to obtain

$$(600+3t)^2X = \frac{135(600+3t)^3}{9} + C.$$

Thus

$$X = 15(600+3t) + \frac{C}{(600+3t)^2}.$$

We can find C at this point by plugging in the initial condition

$$5 = 15(600) + \frac{C}{(600)^2}$$

$$5(600)^2 = 15(600)^3 + C.$$

$$5(600)^2(1-3(600)) = C$$

$$X(t) = 15(600+3t) - \frac{5(600)^2(1799)}{(600+3t)^2}.$$

We conclude that

$$\boxed{X(300) = 15(1500) - \frac{5(600)^2(1799)}{(1500)^2} \text{ pounds of salt.}}$$

[1] In our problem $0 \le t \le 300$; hence $600+3t$ is positive and $|600+3t| = 600+3t$.