Math 242, Exam 1, Spring 2017, 1:15 Class

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators or Cell phones.

(1) **Solve the initial value problem $\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$. Express your answer in the form $y = y(x)$. Please check your answer.**

Separate the variables and integrate:

$$\int y^{-3}dy = \int \frac{xdx}{\sqrt{1+x^2}}$$

$$\frac{y^{-2}}{-2} = \sqrt{1+x^2} + C$$

$$\pm\frac{1}{\sqrt{-2\sqrt{1+x^2} - 2C}} = y$$

Plug in the initial condition to see that the $\pm$ in the front is really $-$ and

$$-\frac{1}{\sqrt{-2\sqrt{1} - 2C}} = -1$$

$$1 = \sqrt{-2\sqrt{1} - 2C}$$

$$-2C = 3.$$

Our solution is

$$\boxed{y = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}.}$$

Check. We see that $y(0) = -\frac{1}{\sqrt{3-2}} = -1\checkmark$. We see that

$$\frac{dy}{dx} = -\frac{\frac{-2(2x)}{2\sqrt{1+x^2}}}{-2(3 - 2\sqrt{1+x^2})^{3/2}} = \frac{-x}{\sqrt{1+x^2}(3 - 2\sqrt{1+x^2})^{3/2}}.$$

On the other hand,

$$\frac{xy^3}{\sqrt{1+x^2}} = \frac{-x}{(3 - 2\sqrt{1+x^2})^{3/2}\sqrt{1+x^2}}.$$

These agree. Our solution is correct.

(2) **The number of bacteria in a liquid culture is observed to grow at a rate proportional to the number of cells present. At the beginning of the experiment there are 10,000 cells and after three hours there are 500,000. How many will there be after one day of growth if this unlimited growth continues? What is the doubling time of the bacteria? Set up an initial value problem. Solve the initial value problem.**

Let $A(t)$ be the number of bacteria in the culture at time t, with t measured in hours. We are told that $\frac{dA}{dt} = kA$, $A(0) = 10,000$, and $A(3) = 500,000$. Separate the variables and integrate:

$$\int \frac{dA}{A} = \int k dt$$

$$\ln A = kt + C$$

$$A = e^C e^{kt}$$

Plug in the initial condition to learn

$$10,000 = A(0) = e^C.$$

So the bacteria population at time t is

$$A(t) = 10,000e^{kt}.$$

We learn k by plugging in $t = 3$

$$500,000 = A(3) = 10,000e^{3k}.$$

$$\frac{\ln 50}{3} = k.$$

$$A(24) = 10,000e^{24\frac{\ln 50}{3}} = 10,000(50)^8$$

$$\boxed{\text{After one day, there are } 10,000(50)^8 \text{ bacteria.}}$$

The population has doubled when $A(t) = 20,000$. We will find this t.

$$20,000 = 10,000e^{kt}$$

$$\ln 2 = kt$$

So,

$$t = \frac{\ln 2}{k} = \frac{3\ln 2}{\ln 50}$$

$$\boxed{\text{The population doubles every } \frac{3\ln 2}{\ln 50} \text{ hours.}}$$

(3) **Consider the initial value problem $\frac{dy}{dx} = x + \frac{1}{y}$, $y(1) = 2$. Use Euler's method to approximate $y(12/10)$. Use two steps, each of size $1/10$.**

Let $f(x,y) = x + \frac{1}{y}$, $(x_0, y_0) = (1,2)$, $x_1 = \frac{11}{10}$, and $x_2 = \frac{12}{10}$. Define y_1 so that the slope of the line joining (x_0, y_0) to (x_1, y_1) is $f(x_0, y_0)$. Define y_2 so that the slope of the line joining (x_1, y_1) to (x_2, y_2) is $f(x_1, y_1)$. Then y_2 is our approximation of $y(\frac{12}{10})$.

$$\frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$$

$$\frac{y_1 - 2}{\frac{1}{10}} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$y_1 = 2 + \left(\frac{1}{10}\right)\left(\frac{3}{2}\right) = 2 + \frac{3}{20} = \frac{43}{20}.$$

$$\frac{y_2 - y_1}{x_2 - x_1} = f(x_1, y_1)$$

$$\frac{y_2 - \frac{43}{20}}{\frac{1}{10}} = \frac{11}{10} + \frac{20}{43}.$$

$$y_2 = \frac{43}{20} + \frac{1}{10}\left(\frac{11}{10} + \frac{20}{43}\right).$$

$$\boxed{\text{Our approximation of } y(\tfrac{12}{10}) \text{ is } y_2 = \tfrac{43}{20} + \tfrac{1}{10}\left(\tfrac{11}{10} + \tfrac{20}{43}\right).}$$

(4) **Find the general solution of $\frac{dy}{dx} + y = e^{-x}$. Express your answer in the form $y = y(x)$. Please check your answer.**

This is a first order linear DE. Multiply both sides of the equation by

$$\mu(x) = e^{\int P(x)dx} = e^{\int 1dx} = e^x$$

and integrate:

$$e^x \frac{dy}{dx} + e^x y = 1.$$

$$e^x y = x + C$$

$$\boxed{y = (x + C)e^{-x}.}$$

<u>Check.</u> Plug our proposed answer into the left side of the DE to get

$$-(x + C)e^{-x} + e^{-x} + (x + C)e^{-x} = e^{-x}.\checkmark$$

(5) **Suppose that a motor boat is moving at 60 feet/second when its motor suddenly quits and that 10 seconds later the boat has slowed to 40 feet/second. Assume that the only force acting on the boat is resistance and resistance is proportional to velocity. How far will the boat travel in all? Set up an initial value problem. Solve the initial value problem.**

Let $x(t)$ equal the position of the boat at time t, where t measures the amount of time since the motor quit. We are told

$$x'' = -kx', \quad x'(0) = 60, \quad x'(10) = 40, \quad \text{and} \quad x(0) = 0,$$

for some positive constant k. If you like, let $v = x'$. Separate the variables in $\frac{dv}{dt} = -kv$ and integrate $\int \frac{dv}{v} = \int -kdt$:

$$\begin{aligned} \ln|v| &= -kt + C \\ |v| &= e^{-kt+C} \\ v &= Ke^{-kt} \\ x' &= Ke^{-kt} \end{aligned}$$

Plug in $t = 0$ to learn $K = 60$. So,

$$x' = 60e^{-kt}$$

Plug in $t = 10$ to learn

$$\begin{aligned} 40 &= 60e^{-10k} \\ \frac{2}{3} &= e^{-10k} \\ \ln\left(\frac{2}{3}\right) &= -10k \\ \frac{\ln\left(\frac{2}{3}\right)}{-10} &= k \end{aligned}$$

Integrate with respect to t to see that

$$x = \frac{60}{-k}e^{-kt} + C_1$$

Plug in $t = 0$ to see that

$$0 = \frac{60}{-k} + C_1$$

so $C_1 = \frac{60}{k}$ and $x(t) = \frac{60}{k} - \frac{60}{-k}e^{-kt}$. We see that x' is never zero; but $\lim\limits_{t\to\infty} x' = 0$. The total distance traveled by the boat is

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{60}{k} - \frac{60}{-k}e^{-kt} = \frac{60}{k} = \frac{60}{\frac{\ln\left(\frac{2}{3}\right)}{-10}} = \frac{-600}{\ln\left(\frac{2}{3}\right)} \text{feet.}$$

$$\boxed{\text{The total distance traveled by the boat is } \frac{-600}{\ln\left(\frac{2}{3}\right)} \text{ feet}}.$$

By the way $\ln(\frac{2}{3})$ is a negative number, so our answer is positive, like it is supposed to be.