Math 242, Exam 1, Spring, 2017 11:40 class, solutions

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators or Cell phones.

(1) **Solve the initial value problem $\frac{dy}{dx} = e^{-y}(2x-4)$, $y(5) = 0$. Express your answer in the form $y = y(x)$. Please check your answer.**

Separate the variables and integrate:

$$\int e^y dy = \int (2x-4)dx$$

$$e^y = x^2 - 4x + C$$

$$1 = e^0 = 25 - 20 + C$$

Plug in the initial condition to evaluate the constant:

$$-4 = C$$

$$e^y = x^2 - 4x - 4$$

$$\boxed{y = \ln(x^2 - 4x - 4)}$$

Check: When $x = 5$, $y = \ln(25 - 20 - 4) = \ln(1) = 0$. ✓Also, $\frac{dy}{dx} = \frac{2x-4}{x^2-4x-4}$ and

$$e^{-y}(2x-4) = (2x-4)e^{-\ln(x^2-4x-4)} = \frac{2x-4}{x^2-4x-4}.$$

Thus our solution works the Differential Equation. ✓

(2) **As part of his summer job at a restaurant, Jim learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. He also found out that, while refrigeration was essential to preserve the soup overnight, the soup was too hot to be put directly into the fridge when it was ready. (The soup had just boiled at 100 degrees C, and the fridge was not powerful enough to accommodate a big pot of soup if it was any warmer than 20 degrees C). Jim discovered that by cooling the pot in a sink full of cold water, (kept running, so that its temperature was roughly constant at 5 degrees C) and stirring occasionally, he could bring the temperature of the soup to 60**

degrees C in ten minutes. How long before closing time should the soup be ready so that Jim could put it in the fridge and leave on time? When you do this problem use Newton's Law of cooling which states that the rate at which an object cools is proportional to the difference between the temperature of the object and the ambient temperature. Set up an initial value problem. Solve the initial value problem.

Let $T(t)$ represent the temperature of the soup at time t. Measure T in degrees C and time in minutes. Start time when the soup is ready. We will find the t that causes $T(t)$ to equal 20. The Differential Equation is $\frac{dT}{dt} = k(T-5)$. The initial condition is $T(0) = 100$. The information $T(10) = 60$ will help us find the constant of proportionality.

Separate the variables and integrate

$$\int \frac{dT}{T-5} = \int k dt$$

$$\ln(T-5) = kt + C$$

$$T - 5 = e^C e^{kt}$$

Plug in the initial condition to evaluate the constant:

$$100 - 5 = e^C e^0$$

We now know that the temperature of the soup at time t is:

$$T(t) = 5 + 95e^{kt}.$$

Plug in the information that tells us k:

$$60 = T(10) = 5 + 95e^{10k}$$

$$\frac{55}{95} = e^{10k}$$

$$\frac{\ln(\frac{55}{95})}{10} = k.$$

The soup has cooled to 20 degrees when

$$20 = T(t) = 5 + 95e^{kt}$$

$$\frac{\ln(\frac{15}{95})}{k} = t$$

$$10\frac{\ln(\frac{15}{95})}{\ln(\frac{55}{95})} = t$$

$$\boxed{\text{The soup should be ready } 10\frac{\ln(\frac{15}{95})}{\ln(\frac{55}{95})} \text{ minutes before closing time.}}$$

(3) (a) **State the Existence and Uniqueness Theorem for first order Differential Equations.**

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. If $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are both continuous on some open neighborhood of (x_0, y_0), then the initial value problem has a unique solution defined $y = y(x)$, which is defined on some open interval containing x_0.

(b) **What does what does (a) tell you about the Initial Value Problem**

$$(x + y)\frac{dy}{dx} = x^2 + y^2 \quad y(1) = -1?$$

The Existence and Uniqueness Theorem does not apply to the given initial value problem because the "f" is $\frac{x^2+y^2}{x+y}$ which is NOT EVEN DEFINED at $(1, -1)$! (So the Existence and Uniqueness Theorem does not tell us anything about the problem of part (b).)

(4) **Find the general solution of the Differential Equation** $x\frac{dy}{dx} + 2y = x^2 - x + 1$**. Please check your answer.**

This is a first order linear DE

$$(*) \quad y' + \frac{2}{x}y = x - 1 + \frac{1}{x}.$$

Let

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2.$$

Multiply both sides of $(*)$ by x^2 and integrate both sides:

$$x^2y' + 2xy = x^3 - x^2 + x.$$

$$x^2y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C.$$

$$\boxed{y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{C}{x^2}}.$$

Check. When we plug the proposed answer into the left side of the original DE, we obtain

$$x(x/2 - 1/3 - 2Cx^{-3}) + 2(\frac{x^2}{4} - \frac{x}{3} + \frac{1}{2} + \frac{C}{x^2}) = x^2 - x + 1\checkmark.$$

(5) **The acceleration of a car is proportional to the difference between 350 ft/sec and the velocity of the car. If this car can accelerate from 0 to 50 ft/sec in 5 seconds, how long will it take for the car to accelerate from rest to 150 ft/sec? Set up an initial value problem. Solve the initial value problem.**

Let $v(t)$ be the velocity of the car (measured in ft/sec) at time t seconds. We are told that $\frac{dv}{dt} = k(350 - v)$. The initial condition is $v(0) = 0$. We are told that $v(5) = 50$. (This allows us to find k.) We are asked to find the time with $v(t) = 150$. We integrate $\int \frac{dv}{350-v} = \int k dt$ to see that

$$-\ln(350 - v) = kt + C \tag{1}$$

The initial condition $v(0) = 0$ tells us that $-\ln(350) = C$. We plug in $v(5) = 50$ into (1) to see that $-\ln(350 - 50) = 5k - \ln(350)$. It follows that

$$\ln(350) - \ln(300) = 5k$$

$$\ln\left(\frac{350}{300}\right) = 5k;$$

so, $\frac{\ln(\frac{7}{6})}{5} = k$. We now find the time when $v(t) = 150$. Again, we use (1). We solve $-\ln(350 - 150) = kt + C$. We solve $-\ln(200) = (\frac{\ln(\frac{7}{6})}{5})t - \ln(350)$. We see that $t = \frac{\ln(350) - \ln 200}{\frac{\ln(\frac{7}{6})}{5}} = 5\frac{\ln(\frac{350}{200})}{\ln(\frac{7}{6})} = \boxed{5\frac{\ln(\frac{7}{4})}{\ln(\frac{7}{6})}\text{sec}}$.