Math 242, Exam 1, Fall 2016

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today. The quiz on Thursday will be one problem from this exam or one of the assigned homework problems from section 1.6.

No Calculators or Cell phones.

(1) **Solve the initial value problem** $2yy' = \dfrac{x}{\sqrt{x^2-16}}$**,** $y(5) = 2$**. Write your answer in the form** $y = y(x)$**. Check your answer.**

Separate the variables and integrate:

$$\int 2y dy = \int \frac{x}{\sqrt{x^2-16}} dx$$

$$y^2 = \sqrt{x^2-16} + C.$$

Plug in $x = 5$:

$$4 = \sqrt{9} + C.$$

So, $C = 1$, $y^2 = \sqrt{x^2-16} + 1$, and $y = \pm\sqrt{\sqrt{x^2-16}+1}$. However, y is a function and $y(5)$ is positive; therefore, $\boxed{y = +\sqrt{\sqrt{x^2-16}+1}}$.

<u>Check.</u> Observe that

$$y(5) = +\sqrt{\sqrt{25-16}+1} = \sqrt{\sqrt{9}+1} = \sqrt{3+1} = \sqrt{4} = 2.\checkmark$$

Also,

$$2yy' = 2\sqrt{\sqrt{x^2-16}+1}\frac{1}{2\sqrt{\sqrt{x^2-16}+1}}\frac{2x}{2\sqrt{x^2-16}} = \frac{x}{\sqrt{x^2-16}}.\checkmark$$

(2) **Solve the initial value problem** $y' + y = e^x$**,** $y(0) = 1$**. Write your answer in the form** $y = y(x)$**. Check your answer.**

This is a first order linear Differential Equation. The equation is already in the form $y' + P(x)y = Q(x)$. We multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int dx} = e^x$$

to obtain:

$$e^x y' + e^x y = e^{2x}.$$

Integrate both sides with respect to x:

$$e^x y = \frac{1}{2}e^{2x} + C.$$

Plug $x = 0$ into the equation:

$$1 = \frac{1}{2} + C;$$

thus $C = \frac{1}{2}$ and

$$\boxed{y = \frac{e^x + e^{-x}}{2}.}$$

<u>Check.</u> Observe that $y(0) = \frac{1+1}{2} = 1$. ✓Observe also that

$$y' + y = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = e^x.✓$$

(3) **Find all constants r for which $y(x) = e^{rx}$ is a solution of $3y'' + 3y' - 4y = 0$.**

Take derivatives of $y = e^{rx}$: $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Plug y into the Differential Equation:

$$3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0$$
$$e^{rx}(3r^2 + 3r - 4) = 0$$

The function e^{rx} is never zero; so,

$$3r^2 + 3r - 4 = 0.$$

Use the quadratic formula to conclude that

$$r = \boxed{\frac{-3 \pm \sqrt{9 + 48}}{6}.}$$

(4) **A motor boat is moving at 40 feet per second when its motor suddenly quits and 10 seconds later the boat has slowed to 20 feet/second. The only force acting on the boat is resistance and resistance is proportional to velocity. How far will the boat coast in all?**

Let $x(t)$ equal the position of the boat at time t, where t measures the amount of time since the motor quit. We are told

$$x'' = -kx', \quad x'(0) = 40, \quad x'(10) = 20, \quad \text{and} \quad x(0) = 0,$$

for some positive constant k. If you like, let $v = x'$. Separate the variables in $\frac{dv}{dt} = -kv$ and integrate $\int \frac{dv}{v} = \int -kdt$:

$$\ln|v| = -kt + C$$
$$|v| = e^{-kt+C}$$

$$v = Ke^{-kt}$$
$$x' = Ke^{-kt}$$

Plug in $t = 0$ to learn $K = 40$. So,

$$x' = 40e^{-kt}$$

Plug in $t = 10$ to learn

$$20 = 40e^{-10k}$$
$$\frac{1}{2} = e^{-10k}$$
$$-\ln 2 = -10k$$
$$\frac{\ln 2}{10} = k$$

Integrate with respect to t to see that

$$x = \frac{40}{-k}e^{-kt} + C_1$$

Plug in $t = 0$ to see that

$$0 = \frac{40}{-k} + C_1$$

so $C_1 = \frac{40}{k}$ and $x(t) = \frac{40}{k} - \frac{40}{-k}e^{-kt}$. We see that x' is never zero; but $\lim_{t\to\infty} x' = 0$. The total distance traveled by the boat is

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{40}{k} - \frac{40}{-k}e^{-kt} = \frac{40}{k} = \frac{40}{\frac{\ln 2}{10}} = \boxed{\frac{400}{\ln 2}\text{feet}}.$$

(5) **A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos t)$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank at time t? Set up the initial value problem. You do not have to solve it.**

Let $x(t)$ equal the number of pounds of salt in the tank at time t. We are told that $x(0) = 5$. We are also told that

$$\frac{dx}{dt} = \frac{1}{5}(1 + \cos t)\frac{\text{lbs}}{\text{gal}}9\frac{\text{gal}}{\text{hr}} - \frac{x}{600 + 3t}\frac{\text{lbs}}{\text{gal}}6\frac{\text{gal}}{\text{hr}}.$$

We must solve the initial value problem:

$$\boxed{\begin{cases} \frac{dx}{dt} = \frac{9}{5}(1 + \cos t) - \frac{6x}{600+3t} \\ x(0) = 5. \end{cases}}$$