MATH 242, SPRING 2025

CONTENTS

1. **Introductory remarks about the course.**

Here are some preliminary remarks about the course.

(1) My name is Professor Kustin.
(2) Be sure to look at the class website often. If you don't know the address, send an e-mail to me at kustin@math.sc.edu
(3) Quiz 1 on Wednesday, January 15 is one of the assigned HW problems from 1.1.
(4) There is a list of assigned HW on my website. (The problems also are on the website.)
(5) The HW is from Differential Equations and Boundary Value Problems computing and modeling sixth edition by Edwards, Penney, and Calvis.
(6) There will be an Exam or Quiz essentially every class. The exams and quizzes will be given at the end of class. When you finish your quiz or exam, **take a picture of your solution for your records** and give me your answers.
(7) I have posted a typed version of the class lectures on my website. I encourage you to study them. In particular, if you miss a lecture, then I strongly encourage you to study the lecture notes. I will revise the typed notes as the class progresses.
(8) Ask about the things you don't understand.
    - Ask in class.
    - Ask during office hours.
    - Send me an e-mail.
    - Catch me before class.
    - Cach me after class.
    - Ask until you are satisfied.
(9) I want you to learn the material. (If I am not happy with your work, I will complain vigorously.) I want you to earn a good grade. (If you mess something up on a quiz or exam, I will surely ask you about it again. Get it right the second time (or the third time). The grading scheme is structured so that the early miss will not harm your final grade if you eventually figure it out. See "How your final grade will be calculated" at the end of the syllabus for full details.)

2. Section 1.1: Words.

Definition. A Differential Equation is an equation that involves a function, its independent variables, and its derivatives.

Examples. The following equations are Differential Equations:

(a) $y'' + y = 0$ (where $y = y(x)$),
(b) $y' = x$ (where $y = y(x)$), and
(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (where $u = u(x, y)$).

Definition. A solution of a Differential Equation is a **FUNCTION** which satisfies the Differential equation.

Examples. (a) All functions of the form $y = a \sin x + b \cos x$ are solutions of

$$y'' + y = 0,$$

where a and b are constants. Indeed, if $y = a \sin x + b \cos x$, then

$$y' = a \cos x - b \sin x \quad \text{and} \quad y'' = -a \sin x - b \cos x.$$

Observe that

$$y'' + y = (-a \sin x - b \cos x) + (a \sin x + b \cos x) = 0.$$

(b) All functions of the form $y = \frac{x^2}{2} + c$ are solutions of $y' = x$, where c is a constant. Indeed, if $y = \frac{x^2}{2} + c$, then $y' = x$.
(c) All functions of the form $u = e^{ax} \cos(ay)$ are solutions of $u_{xx} + u_{yy} = 0$, where a is a constant. Indeed, if $u = e^{ax} \cos(ay)$, then

$$u_x = ae^{ax} \cos(ay), \quad u_{xx} = a^2 e^{ax} \cos(ay),$$

$$u_y = -ae^{ax} \sin(ay), \quad \text{and} \quad u_{yy} = -a^2 e^{ax} \cos(ay).$$

It is now clear that

$$u_{xx} + u_{yy} = (a^2 e^{ax} \cos(ay)) + (-a^2 e^{ax} \cos(ay)) = 0.$$

Definition. The order of a Differential Equation is the largest number of derivatives that appear in the Differential Equation.

So, $y'' + y = 0$ and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ are second order DEs and $y' = x$ is a first order DE.

In Chapter 1, we study first order DEs.

Definition. If the function in a DE involves one variable, then the DE is an Ordinary Differential Equation (ODE). If the function in a DE involves more than one variable, then the DE is a Partial Differential Equation (PDE).

In Math 242, we study ODEs. PDEs are studied in Math 521.

Example. Verify that $y = x\cos(\ln x)$ is a solution of
$$x^2y'' - xy' + 2y = 0.$$

Solution: We calculate
$$y' = \big(x(-\sin(\ln x))\big)\frac{1}{x} + \cos(\ln x) = -\sin(\ln x) + \cos(\ln x)$$
and
$$y'' = -\big(\cos(\ln x)\big)\frac{1}{x} - \big(\sin(\ln x)\big)\frac{1}{x}.$$
We plug y'', y', and y into the left side of the original DE to obtain
$$x^2\Big(-\big(\cos(\ln x)\big)\frac{1}{x} - \big(\sin(\ln x)\big)\frac{1}{x}\Big) - x\Big(-\sin(\ln x) + \cos(\ln x)\Big) + 2\Big(x\cos(\ln x)\Big)$$
$$= \Big(-x\cos(\ln x) - x\sin(\ln x)\Big) + \Big(+x\sin(\ln x) - x\cos(\ln x)\Big) + 2\Big(x\cos(\ln x)\Big).$$
This is zero, as claimed.

Example. Solve the Initial Value Problem[1]
$$y'' + y = 0, \quad y(0) = 3, \quad y'(0) = 4.$$

Solution: To do this problem you have to know the general solution of $y'' + y = 0$. We officially cover problems like this in Chapter 3. But when we get to Chapter 3; the lecture is: the general solution of $y'' + y = 0$ is $y = a\sin x + b\cos x$. (See for the example on page 3.) We may as well just know that now. At any rate, once we decide that $y = a\sin x + b\cos x$ is the general solution of $y'' + y = 0$, then all we have to do is find a and b so that $y(0) = 3$ and $y'(0) = 4$. Well, $y = a\sin x + b\cos x$; so $3 = y(0) = b$. Also, $y' = a\cos x - b\sin x$; so $4 = y'(0) = a$ and $a = 4$. We conclude that
$$\boxed{y = 4\sin x + 3\cos x}$$
is the solution to the IVP.

Check. We calculate
$$y' = 4\cos x - 3\sin x \quad \text{and} \quad y'' = -4\sin x - 3\cos x.$$
Observe that
$$y + y'' = \Big(4\sin x + 3\cos x\Big) + \Big(-4\sin x - 3\cos x\Big) = 0, \checkmark$$
$$y(0) = 4\sin(0) + 3\cos(0) = 3\checkmark \quad \text{and} \quad y'(0) = 4\cos(0) - 3\sin(0) = 4.\checkmark$$

[1] An Initial Value Problem (IVP) is a DE together with the appropriate Initial Condition. A well posed IVP will have a unique solution. This is the main problem we consider in Math 242.

Example. Find all values of r such that $y = e^{rx}$ is a solution of $y'' - 2y' - 3y = 0$.

Solution:[2] We compute that if $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2e^rx$.

If $y = e^{rx}$ is a solution of $y'' - 2y' - 3y = 0$, then

$$r^2e^{rx} - 2re^{rx} - 3e^{rx} = 0.$$

Factor out the e^{rx}. The most recent equation is

$$e^{rx}(r^2 - 2r - 3) = 0.$$

If the product of two numbers is zero, then one of the numbers is zero. But e^{rx} is never zero. Thus, $r^2 - 2r - 3 = 0$. That is $(r-3)(r+1) = 0$. We have learned that if $y = e^{rx}$ is a solution of the DE $y'' - 2y' - 3y = 0$, then

$$\boxed{r = 3 \quad \text{or} \quad r = -1.}$$

Check. If $y = e^{3x}$, then $y' = 3e^{3x}$ and $y'' = 9e^{3x}$ and when these functions are plugged into $y'' - 2y' - 3y$, then the result is

$$9e^{3x} - 2(3e^{3x}) - 3(e^{3x}) = (9 - 6 - 3)e^{3x}$$

and this is zero. In a similar manner, if $y = e^{-x}$, then $y' = -e^{-x}$ and $y'' = e^{-x}$ and when these functions are plugged into $y'' - 2y' - 3y$, then the result is

$$e^{-x} - 2(-e^{-x}) - 3(e^{-x}) = (1 + 2 - 3)e^{-x}$$

and this is zero.

Example 2.1. Find and solve a DE which describes a function $y = g(x)$ with the property that the line tangent to $y = g(x)$ at (x_0, y_0) intersects the x-axis at $(\frac{x_0}{2}, 0)$ for all points (x_0, y_0) on $y = g(x)$.

Solution: First we find the DE. We have two ways to calculate the slope of the line tangent to $y = g(x)$ at (x_0, y_0).

On the one hand, this slope must be $g'(x_0)$.

On the other hand, this tangent line passes through the points (x_0, y_0) and $(\frac{x_0}{2}, 0)$. Thus the slope must equal

$$\frac{y_0 - 0}{x_0 - \frac{x_0}{2}} = \frac{y_0}{\frac{x_0}{2}} = \frac{2y_0}{x_0}.$$

Thus,

$$g'(x_0) = \frac{2y_0}{x_0}$$

for all points (x_0, y_0) which satisfy the equation $y = g(x)$. The DE which describes our curve is

$$y' = \frac{2y}{x}.$$

[2]This is the basic technique of Chapter 3.

To solve the DE we separate the variables[3] getting all of the x's and dx's on one side and all of the y's and dy's on the other side. Then we integrate both sides.

The DE can be written as $\frac{dy}{dx} = \frac{2y}{x}$. Multiply both sides by dx and divide both sides by y to obtain

$$\frac{dy}{y} = \frac{2}{x}dx.$$

Integrate both sides

$$\int \frac{dy}{y} = 2\int dxx$$
$$\ln|y| = 2\ln|x| + C$$

We want to solve for y; so we exponentiate both sides

$$e^{\ln|y|} = e^{2\ln|x|+C}.$$

Recall that $e^{\ln w} = w$ (for all positive w), $e^{a+b} = e^a e^b$ (for all a and b), and $e^{ab} = (e^a)^b$ (for all a and b). So, the most recent equation is

$$|y| = e^C|x|^2.$$

Of course, $|y| = \pm y$. Multiply both sides by $\pm$. Let $K = \pm e^C$. Notice that $|x|^2 = x^2$. Our solution is

$$\boxed{y = Kx^2 \quad \text{for any non-zero constant } K.}$$

Check. Fix a non-zero constant K and fix a point (x_0, y_0) on $y = Kx^2$. The origin does not work particularly well because the tangent line is the x-axis; so the intersection of the tangent line and the x-axis is more than one point. So we assume ahead of time that x_0 is not zero. We calculate the equation of the line tangent to $y = Kx^2$ at (x_0, y_0) and show that $(\frac{x_0}{2}, 0)$ is a point on the tangent line. The slope of the tangent line is $y'(x_0) = 2Kx_0$. Observe that $y_0 = y(x_0) = Kx_0^2$. The tangent line is $y - y_0 = 2Kx_0(x - x_0)$. Observe that $(x_0/2, 0)$ is on the tangent line because

$$0 - y_0 = 2Kx_0\left(\frac{x_0}{2} - x_0\right)$$

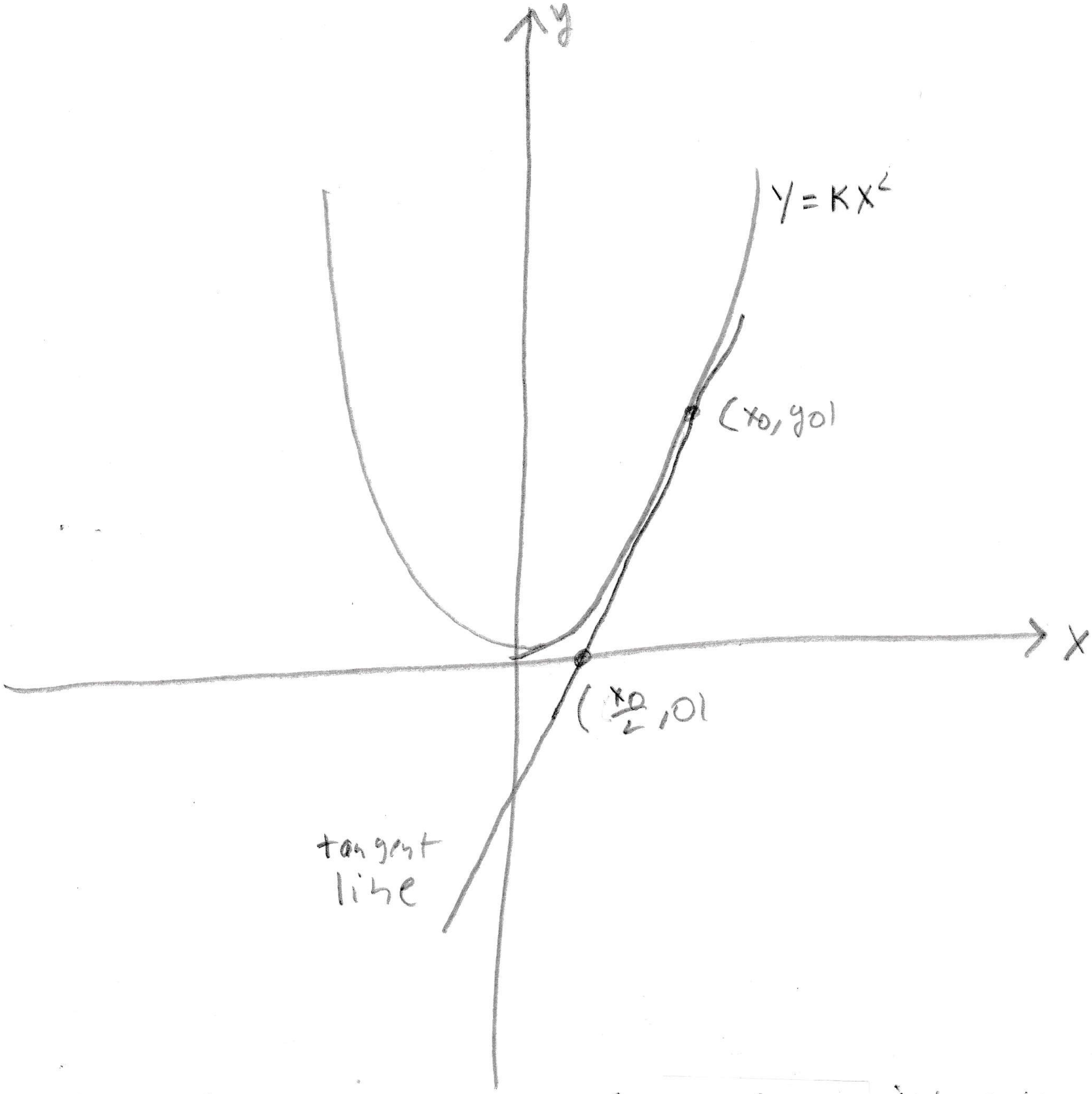
since the right side is

$$2Kx_0\left(-\frac{x_0}{2}\right) = -Kx_0^2 = -y_0,$$

as claimed. There is a picture on the next page.

[3] This is the technique of section 1.4.

Picture 1



The line tangent to $y = kx^2$ at (x_0, y_0) intersects the x-axis at $(\frac{x_0}{2}, 0)$ for all points (x_0, y_0) on the graph of $y = kx^2$ when $x_0 \neq 0$.

3. Section 1.2: Direct integration.

Lecture. To solve $y' = f(x)$, integrate both sides:

$$y = \int f(x)dx.$$

Example. Solve the Initial Value Problem: $y' = \ln x$, $y(1) = 5$.

Solution: Integrate both sides to obtain

$$y = \int \ln x dx.$$

To do this integral we use Integration by Parts:[4]

$$\int u dv = uv - \int v du.$$

Take $u = \ln x$ and $dv = dx$. Compute $du = \frac{1}{x}dx$ and $v = x$. Observe that

$$\int \ln x dx = \int u dv = uv - \int v du = x\ln x - \int dx = x\ln x - x + C.$$

Of course, this is correct because

$$\frac{d}{dx}(x\ln x - x) = x\frac{1}{x} + \ln x - 1 = \ln x.$$

We have found that $y = x\ln x - x + C$ satisfies the DE $y' = \ln x$ for all constants C. Now we must find the C which also causes the Initial Condition to be satisfied. We need

$$5 = y(1) = 1\ln(1) - 1 + C.$$

Of course, $\ln(1) = 0$. We take $C = 6$. The solution is

$$\boxed{y = x\ln x - x + 6.}$$

[4] Integration by parts is a reformulation of the product rule. We use it often. The product rule says $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Multiply both sides by dx to obtain $d(uv) = udv + vdu$; integrate both sides: $uv = \int udv + \int vdu$; and rearrange things to obtain the ultimate formula

$$\int u dv = uv - \int v du.$$

One uses Integration by Parts to integrate the product of unrelated functions (like $\int e^x \sin x dx$), the inverse of well-understood functions (like $\int \ln x dx$ (Notice that $\ln x$ is the inverse of e^x and e^x is exceptionally well-understood.)), and tricky Trig functions (like $\int \sec^3 x dx$).

4. Sections 1.3 and 2.4: The Existence and Uniqueness Theorem and Euler's method.

Section 1.3 is "The Existence and Uniqueness Theorem for First Order Initial Value Problems".

Section 2.4 is "Euler's method for approximating a numerical solution to an Initial Value Problem".

Section 1.3 is the most theoretical section that we cover; section 2.4 is the most grubby section that we cover.

Both sections boil down to the same thing. They both boil down to the fact that

Most of the time the IVP

$$\frac{dy}{dx} = f(x,y) \quad y(x_0) = y_0$$

has a unique solution.

Remarks.

- **This is awesome!** Think how many situations in life do not have a solution!
- Of course, I will make sense of the key word "most".
- The numerical approach explains the theorem!

• Here is Euler's method for approximating a numerical solution for an Initial Value Problem.

Consider the IVP $y' = f(x,y)$, $y(x_0) = y_0$.

The goal[5] is to approximate $y(x_{\text{last}})$.

The method is essentially obvious.

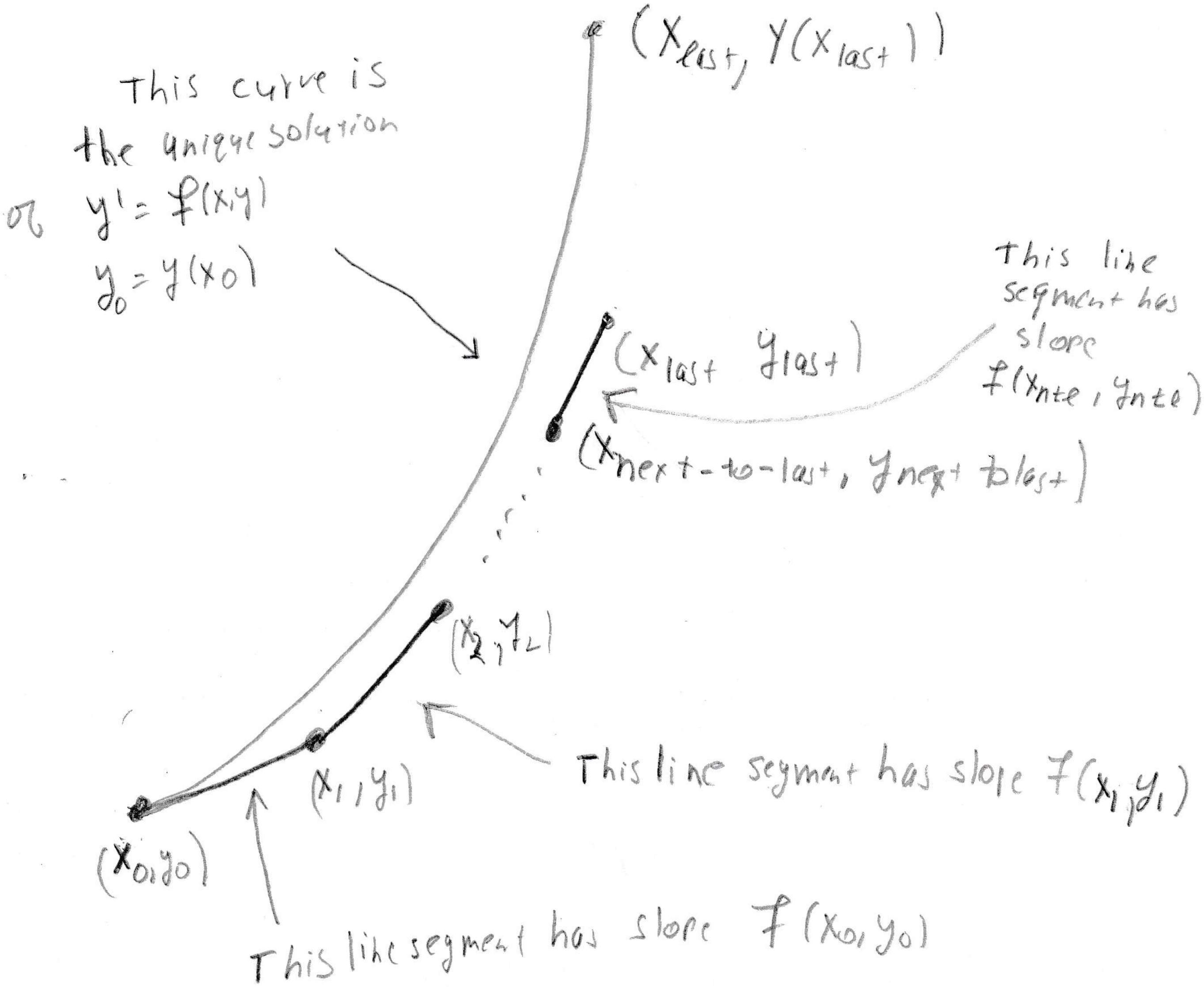
- Start at (x_0, y_0).
- Take a small step along the line through (x_0, y_0) with slope $f(x_0, y_0)$.
- Now you are standing at the point (x_1, y_1). Take a small step along the line through (x_1, y_1) with slope $f(x_1, y_1)$.
- Proceed in this manner until your x-coordinate is x_{last}. Whatever your y-coordinate is is your approximation of $y(x_{\text{last}})$.

If you want a better approximation, take smaller steps. If f is sufficiently continuous, take a limit of your approximations.

There is a picture of Euler's method on the next page.

[5] Of course, the "best" answer would be the entire function y which solves the IVP. But if one does not know how to find all of y, then the next best thing is to approximate y evaluated at the last x of interest.

Euler's Method



y_{last} is the Euler's Method Approximation of $y(x_{last})$

Example 4.1. Consider the Initial Value Problem $y' = 2y$, $y(0) = \frac{1}{2}$.

(a) Use Euler's Method to approximate $y(\frac{1}{2})$. Use two steps. Make each step size be $h = \frac{1}{4}$ in the x-direction.

(b) Solve the IVP. What is the real value of $y(\frac{1}{2})$?

Solution: There is a picture for (a) on the next page. Notice that last $= 2$, $x_0 = 0$, $x_1 = \frac{1}{4}$, and $x_2 = \frac{1}{2}$. The number y_1 is determined by the fact that the line segment joining (x_0, y_0) to (x_1, y_1) has slope $f(x_0, y_0)$ (where $f(x,y) = 2y$). The number y_2 is determined by the fact that the line segment joining (x_1, y_1) to (x_2, y_2) has slope $f(x_1, y_1)$.

The line joining (x_0, y_0) to (x_1, y_1) has slope $f(x_0, y_0)$. We know all of these numbers except y_1; so we can calculate y_1. We know $(x_0, y_0) = (0, \frac{1}{2})$. We know $x_1 = \frac{1}{4}$. We know $f(x_0, y_0) = 2y_0 = 2(\frac{1}{2}) = 1$. Thus,

$$\frac{y_1 - \frac{1}{2}}{\frac{1}{4} - 0} = 1 \quad \text{and} \quad y_1 = \frac{3}{4}.$$

The line joining (x_1, y_1) to (x_2, y_2) has slope $f(x_1, y_1)$. We know all of these numbers except y_2; so we can calculate y_2. We know $(x_1, y_1) = (\frac{1}{4}, \frac{3}{4})$. We know $x_2 = \frac{1}{2}$. We know $f(x_1, y_1) = 2y_1 = 2(\frac{3}{4}) = \frac{3}{2}$. Thus,

$$\frac{y_2 - \frac{3}{4}}{\frac{1}{2} - \frac{1}{4}} = \frac{3}{2} \quad \text{and} \quad y_2 = \frac{9}{8} = \boxed{1.125}$$

(b) To solve the DE $y' = 2y$ we separate the variables, then integrate. The DE is $\frac{dy}{dx} = 2y$, which becomes $\int \frac{dy}{y} = \int 2dx$. Thus, $\ln|y| = 2x + C$.

$$e^{\ln|y|} = e^{2x+C}$$
$$|y| = e^C e^{2x}$$
$$y = \pm e^C e^{2x}$$

Let K be the constant $\pm e^C$. The general solution of the DE $y' = 2y$ is $y = Ke^{2x}$. Now we pick K so that the initial condition $y(0) = \frac{1}{2}$ is satisfied:

$$\frac{1}{2} = y(0) = Ke^0 = K.$$

Thus, the real solution of the initial value problem is $y = \frac{1}{2}e^{2x}$ and the real value of $y(\frac{1}{2})$ is $\boxed{y(\frac{1}{2}) = \frac{e}{2}}$. The number $e/2$ is approximately equal to 1.359. Thus, our approximation $y_2 = \frac{9}{8}$ of $y(\frac{1}{2})$ is close to, but less than, the real value of $y(\frac{1}{2})$.

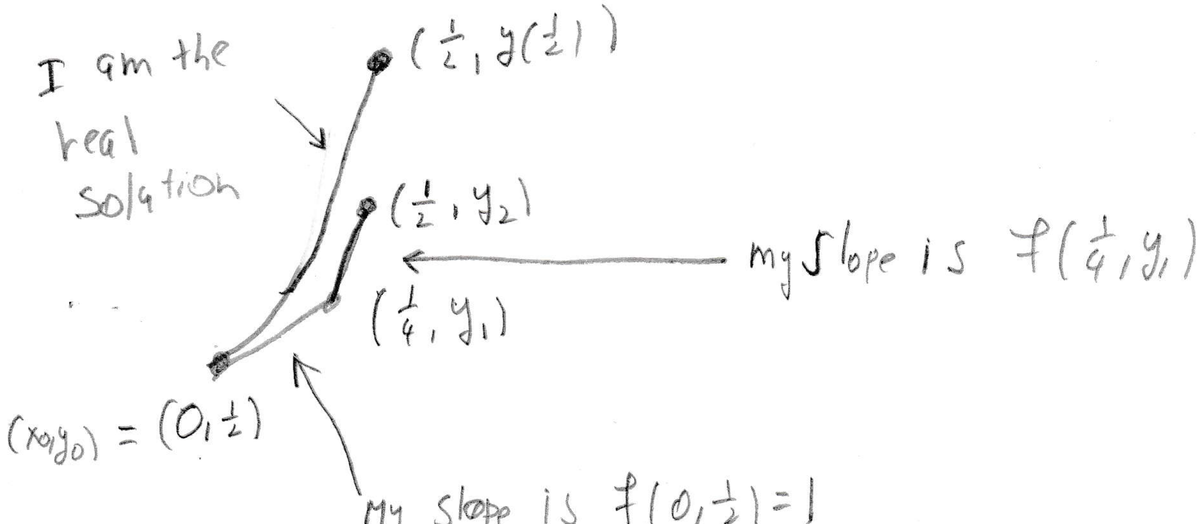
Euler's Method Example

Consider the Initial Value Problem $y' = 2y$

$y(0) = \frac{1}{2}$

Approximate $y(\frac{1}{2})$.

Use 2 steps. Each increment of x should be $\frac{1}{4}$.



y_2 is our approximation of $y(\frac{1}{2})$.

I promised to tell you an honest version of the Existence and Uniqueness Theorem for first order Initial Value Problems.

Theorem. *Assume $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on some open region which contains (x_0,y_0) in its interior. Then the Initial Value Problem*

$$\frac{dy}{dx} = f(x,y) \quad y(x_0) = y_0$$

has a unique solution $y = y(x)$ which is defined on some open interval I which contains x_0.

I put a picture of The Existence and Uniqueness Theorem on the next page.

I should define "continuous". The function $f(x,y)$ is continuous at (x_0,y_0) if

(1) $f(x_0,y_0)$ exists,
(2) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, and
(3) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

The main point is that if $f(x_0,y_0)$ or $\frac{\partial f}{\partial y}(x_0,y_0)$ does not make any sense (because the symbols tell you to divide by zero, or take the square root of a negative number, or take the logarithm of a non-positive number), then the Existence and Uniqueness Theorem does not make any promises.

Keep in mind that all of our favorite functions (polynomials, rational functions, trig functions, exponential functions, logarithm functions, square root functions, etc.) are continuous wherever they are defined.

Example. What does the Existence and Uniqueness say about the IVP

$$y' = 2y, \quad y(0) = \frac{1}{2}? \tag{4.1.1}$$

The "f" is $f(x,y) = 2y$ and $\frac{\partial f}{\partial y} = 2$. (Of course, $x_0 = 0$ and $y_0 = \frac{1}{2}$.) The functions $2y$ and 2 are continuous everywhere. The Existence and Uniqueness Theorem guarantees that the IVP (4.1.1) has a unique solution which is defined on some open interval which contains 0.

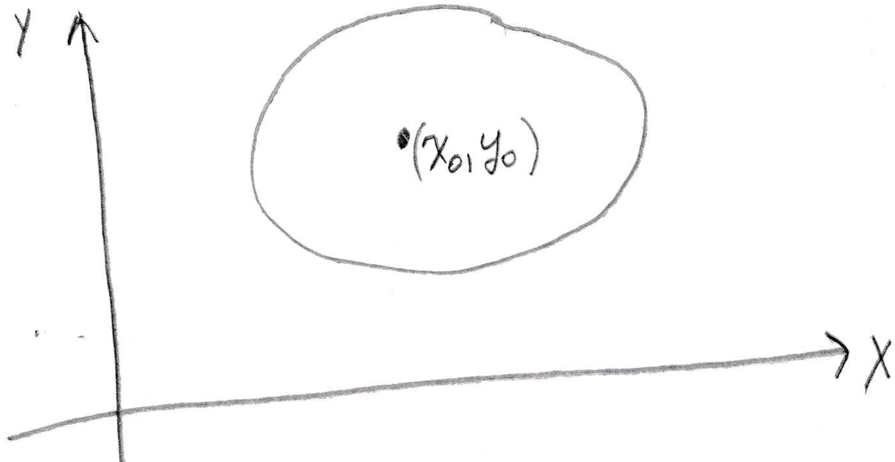
Example. What does the Existence and Uniqueness Theorem say about Initial Value Problems of the form

$$y' = \frac{2y}{x} \quad y(0) = y_0?$$

We thought about this problem in Example 2.1 on page 5. In this problem "f" is $\frac{2y}{x}$ and $\frac{\partial f}{\partial y} = \frac{2}{x}$. Neither f nor $\frac{\partial f}{\partial y}$ is continuous at $(0,y_0)$, so the Existence and Uniqueness Theorem **makes no guarantees here**. (The Theorem is a one way result. If the hypotheses are satisfied, then the conclusion will happen. The Theorem makes no assertion about what happens when the hypotheses are not satisfied.) On the other hand, when we did Example 2.1 we saw that neither our problem nor our answer really made sense when $x = 0$.

A picture version of the Existence and Uniqueness Theorem for first order Initial Value Problems

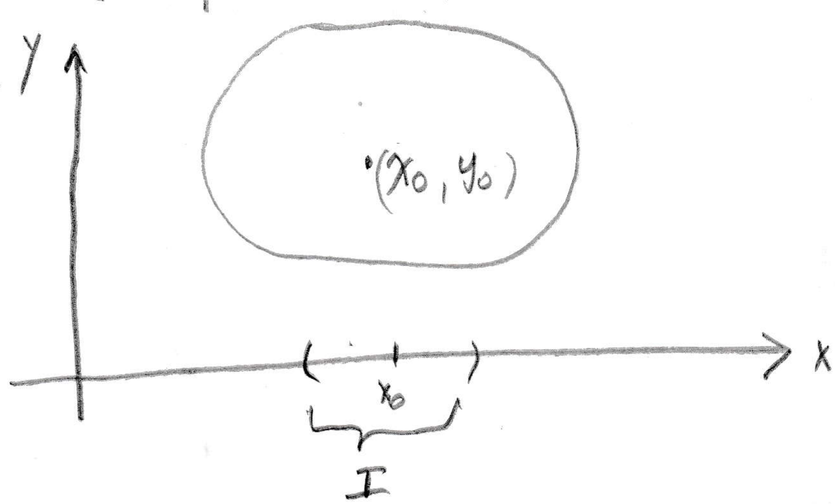
If f and $\frac{\partial f}{\partial y}$ are functions of 2 variables which are well-behaved inside the circle



then the Initial Value Problem

$$\frac{dy}{dx} = f(x,y) \qquad y_0 = y(x_0)$$

has a unique solution $y = y(x)$ which is defined on some open interval I which contains x_0



Section 1.2, problem 32. Suppose a car skids 15 m if it is moving at 50 km/h when the brakes are applied. Assuming that the car has the same constant deceleration, how far will the car skid if it is moving at 100 km/hr when the brakes are applied?
Solution: Let x be the position of the car at time t. Let $t = 0$ be the moment the brakes are applied. Take $x(0) = 0$. Let $-k$ be the constant deceleration of this car. (Notice that k is a positive number.) Let v_0 be the speed of the car when the brakes are applied. Our plan is to find a formula for the stopping distance in terms of k and v_0. Let t_{stop} be the time when the car stops. Of course, $x(t_{\text{stop}})$ is how far the car skidded.

The Initial Value Problem is $x'' = -k$, $x'(0) = v_0$, and $x(0) = 0$. The solution of the IVP is $x(t) = -kt^2/2 + v_0t$ and $x'(t) = -tk + v_0$. The car stops when $x'(t_{\text{stop}}) = 0$; so $0 = -kt_{\text{stop}} + v_0$. So the car stops at $t_{\text{stop}} = \frac{v_0}{k}$. The position of the car when it stops is $x(t_{\text{stop}}) = -k(\frac{v_0}{k})^2/2 + v_0(\frac{v_0}{k}) = \frac{v_0^2}{2k}$. **If the initial speed is doubled then the stopping distance is multiplied by a factor of 4.** In particular if the initial speed is raised from 50 k/h to 100 k/h, then the stopping distance increases from 15 m to 60 m.

5. Sections 1.4 and 2.3: Separate the variables.

Here is the lecture for section 1.4: If it is possible to get all of the x's and dx's on one side and all of the y's and dy's on the other side, then do it and integrate both sides.

Section 2.3 is called Motion problems; but it turns out that all of the problems in the homework set for 2.3 can be solved by separating the variables. We solved Example 2.1 by separating the variables. We solved Example 4.1.(b) by separating the variables. We solved homework problem 46 from section 1.1 by separating the variables.

Example. (This is problem 46 from section 1.1.) Suppose the velocity v of a motorboat coasting in the water satisfies the DE $\frac{dv}{dt} = kv^2$. The initial velocity of the motorboat is $v(0) = 10$ meter/second^2 and v is decreasing at the rate of 1 m/s^2, when $v = 5$ m/s.[6] How long does it take for the velocity of the boat to decrease to 1 m/s? to $1/10$ m/s? When does the boat come to a stop?

[6] Keep in mind that $\frac{dv}{dt}$ is acceleration and Newton's Second Law of motion is $F = ma$. The DE says that the force acting on the boat is proportional to the velocity of the boat. The velocity is decreasing. So the force acting on the boat is a resisting force. It appears plausible to think that when the boat is going fast, the there is a great deal of resistance because the water is slapping really hard against the side of the boat. But when the boat slows down there will be less slapping and less resistance. So at first glance $\frac{dv}{dt} = kv^2$ appears to be a reasonable DE for the motion of this boat. Stay tuned!

First we record what we know:

$$\frac{dv}{dt} = kv^2$$
$$v(0) = 10$$
$$\left.\frac{dv}{dt}\right|_{v=5} = -1$$

Of course, the questions are:

(a) Find t with $v(t) = 1$.
(b) Find t with $v(t) = \frac{1}{10}$.
(c) Find t with $v(t) = 0$.

Notice that

$$\frac{dv}{dt} = kv^2$$
$$v(0) = 10$$

is an ordinary IVP.

The information $\frac{dv}{dt}|_{v=5} = -1$ is given so that we can evaluate k. Indeed, when we plug this information into $\frac{dv}{dt} = kv^2$, we learn that $-1 = k(25)$; so, $\frac{-1}{25} = k$. (We will plug this in at the end. It is easier to deal with "k" than to deal with a grubby number.)

It is easy to separate the variables in $\frac{dv}{dt} = kv^2$; namely

$$\int \frac{dv}{v^2} = \int kdt$$
$$-v^{-1} = kt + C$$

We solve for v

$$\frac{-1}{kt+C} = v.$$

Use the initial condition to find C:

$$\frac{-1}{C} = v(0) = 10.$$

So, $C = -\frac{1}{10}$ and

$$\frac{-1}{\frac{-1}{25}t - \frac{1}{10}} = v.$$

Multiply top and bottom by -50 to learn

$$v = \frac{50}{2t+5}.$$

Now we answer (a). We see that $v(t) = 1$ when

$$1 = \frac{50}{2t+5}$$
$$2t + 5 = 50$$

$$2t = 45$$

$$\boxed{t = 22.5 \text{ seconds.}}$$

We answer (b). We see that $v(t) = \frac{1}{10}$ when

$$\frac{1}{10} = \frac{50}{2t+5}$$
$$2t+5 = 500$$
$$2t = 495$$
$$\boxed{t = 247.5 \text{ seconds.}}$$

We answer (c). We see that $v(t) = 0$ when

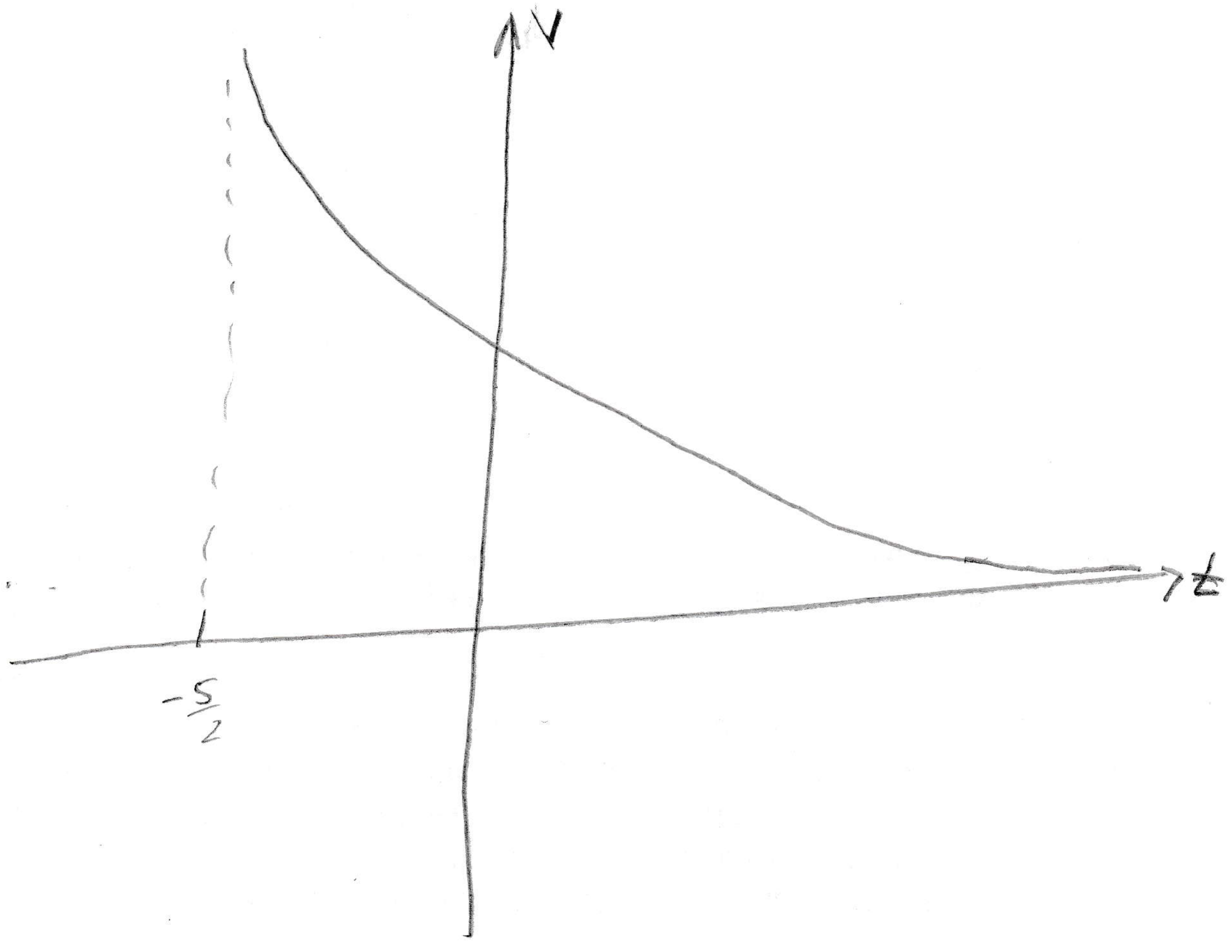
$$0 = \frac{50}{2t+5}$$

The boat stops when $0 = 50$. Oh darn.

$$\boxed{\text{The boat never stops}}.$$

I put a picture of $v(t) = \frac{50}{2t+5}$ on the next page. The t-axis is a horizontal asymptote for the graph. The graph, indeed, never gets to $v = 0$.

The graph of $v(t) = \dfrac{50}{2t+5}$



The velocity never reaches zero.

$$\frac{dv}{dt} = kv^2$$

is not a good model for motion when v is near 0.

Of course, the boat will come to the end of the lake, or the person in the boat will hop out and pull the boat in to shore, or we will use a different DE when v is small.

Example 5.1. (This is problem 45 from section 1.1.) Let $P(t)$ represent the number of rodents at time t. Suppose P satisfies the differential equation $\frac{dP}{dt} = kP^2$. Initially, there are $P(0) = 2$ rodents and their number is increasing at the rate of $\frac{dP}{dt} = 1$ rodent per month when there are $P = 10$ rodents. How long will it take for the population to grow to 100 rodents? To 1000 rodents? What is happening here?

Answer: The Initial Value Problem is

$$\tfrac{dP}{dt} = kP^2 \quad P(0) = 2.$$

The information "their number is increasing at the rate of $\frac{dP}{dt} = 1$ rodent per month when there are $P = 10$ rodents" is provided to enable us to determine the constant k. When this information is written as an equation it becomes

$$\left.\tfrac{dP}{dt}\right|_{P=10} = 1.$$

At any rate, when we plug $P = 10$ and $\frac{dP}{dt} = 1$ into the equation $\frac{dP}{dt} = kP^2$, we learn that $1 = k100$; so $\frac{1}{100} = k$. I see no reason to use this fact until the end of the calculation.

Now we solve the IVP. First we solve the DE: $\frac{dP}{dt} = kP^2$. We are able to separate the variables (get all of the P's and dP's on one side and all of the t's and dt's on the other side). So we separate the variables and integrate:

$$\int \tfrac{dP}{P^2} = \int k dt$$

$$-\tfrac{1}{P} = kt + C.$$

We need to solve for P. We also need to find C. We may as well find C now while it is still easy to reach (rather than wait until it is thoroughly mixed into things). The initial condition says that when $t = 0$, then $P = 2$; so $-\frac{1}{2} = C$. Again I will keep using C for a while.

We solve for P: $\frac{-1}{kt+C} = P$. Plug $C = -\frac{1}{2}$ and $k = \frac{1}{100}$ into our expression for C to obtain

$$P = \frac{-1}{\frac{1}{100}t - \frac{1}{2}}.$$

Multiply top and bottom by 100:

$$P = \frac{-100}{t-50} \quad \text{or} \quad P = \frac{100}{50-t}.$$

We have solved the IVP. Now we can answer the questions.

The population reaches $P = 100$ when

$$\frac{100}{50-t} = 100$$

so $1 = 50 - t$ and $\boxed{t = 49 \text{ months}}$.

The population reaches $P = 1000$ when

$$\frac{100}{50-t} = 1000$$

so $\frac{1}{10} = 50 - t$ and $\boxed{t = 49.9 \text{ months}}$.

The rat population is exploding! The function $P = \frac{-100}{t-50}$ has a vertical asymptote at $t = 50$ and $\lim_{t \to 50^-} P(t) = +\infty$.

5.A. **Here are three famous and important DEs that are solved by separating the variables.**

(a) **Radioactive Decay** If $A(t)$ is the amount of radioactive material present at time t, then $\frac{dA}{dt} = kA$. See Homework problem 35.

(b) **Continuously compounded interest** If $A(t)$ is the value of an investment at time t, then $\frac{dA}{dt} = kA$. See Homework problem 37.

(c) **Newton's Law of Cooling** If $T(t)$ is the temperature of an object at time t and A is the temperature of the surrounding medium, then $\frac{dT}{dt} = k(T - A)$. See Homework problem 43.

6. Section 1.5: First order linear Differential Equations

A Differential Equation of the form

$$y' + P(x)y = Q(x) \tag{6.0.1}$$

is a First order linear Differential Equation. The equation is called linear because it is linear in y' and y. In other words, some terms have y' **to the one or zero** and the rest of the terms have y **to the one or zero** and that is it! Terms like y^2 or y'^2 or yy' or $\sin y$ or e^y or $\frac{y}{y'}$ **DO NOT APPEAR.**

We can find an explicit solution to a first order linear differential equation, **but there is a trick involved.** The trick is: if you multiply the left side of (6.0.1) by

$$\mu(x) = e^{\int P(x)dx},$$

then you are able to integrate the new left hand side. Indeed, the new left side

$$e^{\int P(x)dx}y' + e^{\int P(x)dx}P(x)y$$

is the derivative of

$$\mu(x)y$$

because the derivative of the product

$$\mu(x)y = e^{\int P(x)dx}y$$

is the first times the derivative of the second plus the second times the derivative of the first

$$e^{\int P(x)dx}y' + ye^{\int P(x)dx}P(x),$$

where the first is $e^{\int P(x)dx}$, the second is y, the derivative of the second is (of course) y' and the derivative of the first is

$$e^{\int P(x)dx}(\text{times the derivative of } \int P(x)dx)$$
$$= e^{\int P(x)dx}P(x).$$

I write μ for Magic Multiplier (μ is a Greek m); other people write $I(x)$ in place of $\mu(x)$ and call $I(x)$ an integrating factor.

Here is an example.

Example. Solve $2xy' - 3y = 9x^3$.

We see that the problem is linear in y and y'. We divide both sides by $2x$ to get the problem into the proper form.

$$y' - \frac{3}{2x}y = \frac{9}{2}x^2 \tag{6.0.2}$$

The DE has the form $y' + P(x)y = Q(x)$ where $P(x) = -\frac{3}{2x}$ and $Q(x) = \frac{9}{2}x^2$.

Multiply both sides of (6.0.2) by

$$\mu(x) = e^{\int P(x)dx} = e^{\int -\frac{3}{2x}dx} = e^{-\frac{3}{2}\ln x} = x^{-\frac{3}{2}} :$$

$$x^{-\frac{3}{2}}y' - x^{-\frac{3}{2}}\tfrac{3}{2x}y = x^{-\frac{3}{2}}\tfrac{9}{2}x^2$$

$$x^{-\frac{3}{2}}y' - \tfrac{3}{2}x^{-\frac{5}{2}}y = \tfrac{9}{2}x^{1/2} \tag{6.0.3}$$

Notice that the left side of (6.0.3) is the derivative of $x^{-\frac{3}{2}}y$ with respect to x. So, we can integrate both sides with respect to x to obtain

$$x^{-\frac{3}{2}}y = \int \frac{9}{2}x^{1/2}dx = \frac{2}{3}\frac{9}{2}x^{3/2} + C = 3x^{3/2} + C.$$

Multiply both sides of the equation by $x^{3/2}$ to see that

$$\boxed{y = 3x^3 + Cx^{3/2}}.$$

Check. Of course, our solution is correct. When we plug it into $2xy' - 3y$, we obtain

$$2x(9x^2 + \tfrac{3}{2}Cx^{1/2}) - 3(3x^3 + Cx^{3/2}) = 18x^3 + 3Cx^{3/2} - 9x^3 - 3Cx^{3/2} = 9x^3,$$

as expected ✓.

Example. This is called a solution problem. It is number 36 in section 1.5. A tank initially contains 60 gal. of pure water. Brine containing 1 lb. of salt per gallon enters the tank at 2 gal./min., and the (perfectly mixed) solution leaves the tank at 3 gal./min.; the tank is empty after exactly one hour.

(a) Find the amount of salt in the tank after t minutes.
(b) What is the maximum amount of salt ever in the tank?

ANSWER: Let $X(t)$ be the number of pounds of salt at time t minutes. It is clear that $\frac{dX}{dt}$ is the rate at which salt enters the tank minus the rate at which salt leaves the tank. It is also clear that salt enters the tank at

$$1\,\tfrac{\text{lb}}{\text{gal}} \times 2\,\tfrac{\text{gal}}{\text{min}} = 2\,\tfrac{\text{lb}}{\text{min}}.$$

The rate at which salt leaves the tank is a little more complicated. It is

$$\text{some number}\tfrac{\text{lb}}{\text{gal}} \times 3\,\tfrac{\text{gal}}{\text{min}}$$

but we have to write something better than "some number". At time t we know how much liquid is in the tank, namely $60 - t$. (We found the equation of the line that passes through $(0,60)$ and losses one gallon per minute.) We have a name for the number of pounds of salt in the tank at time t; namely $X(t)$. So the rate at which salt leaves the tank is

$$\tfrac{X(t)}{60-t}\,\tfrac{\text{lb}}{\text{gal}} \times 3\,\tfrac{\text{gal}}{\text{min}} = \tfrac{3X}{60-t}\,\tfrac{\text{lb}}{\text{min}}$$

Now we have our Initial Value Problem:

$$\tfrac{dX}{dt} = 2 - \tfrac{3X}{60-t} \qquad\qquad X(0) = 0.$$

We solve the IVP. Then we answer the questions.

It is not possible to separate the variables, but we do have a First Order linear DE:

$$\frac{dX}{dt} + \frac{3}{60-t}X = 2. \tag{6.0.4}$$

Take

$$\mu(t) = e^{\int \frac{3}{60-t}dt} = e^{-3\ln(60-t)} = (60-t)^{-3}.$$

(We do not have to write absolute value. We know that this problem all takes place when $0 \leq t \leq 60$, so $60-t$ is zero or higher.) Multiply both sides of (6.0.4) by $(60-t)^{-3}$:

$$(60-t)^{-3}\left(\frac{dX}{dt} + \frac{3}{60-t}X\right) = 2(60-t)^{-3}$$

$$(60-t)^{-3}\frac{dX}{dt} + 3(60-t)^{-4}X = 2(60-t)^{-3} \tag{6.0.5}$$

Notice that the left side of (6.0.5) is

$$\tfrac{d}{dt}((60-t)^{-3}X).$$

Integrate both sides of (6.0.5) with respect to t to obtain

$$(60-t)^{-3}X = \int 2(60-t)^{-3}dt = (60-t)^{-2} + C.$$

Multiply both sides of the most recent equation by $(60-t)^3$ to obtain

$$X = (60-t) + C(60-t)^3.$$

Use the Initial Condition $X(0) = 0$ to see that

$$0 = (60) + C(60)^3.$$

Thus, $C = \frac{-1}{(60)^2}$ and

$$\boxed{X(t) = (60-t) - \frac{1}{(60)^2}(60-t)^3}$$

is the answer to (a).

Question (b) is a calculus problem. We are to find the maximum of $X(t)$ for $0 \leq t \leq 60$. We know $X(0) = X(60) = 0$. So the maximum of $X(t)$ occurs when $X'(t) = 0$. We compute $X'(t) = -1 + \frac{3}{(60)^2}(60-t)^2$. We see that $X'(t) = 0$ when

$$\begin{aligned} 1 &= \tfrac{3}{(60)^2}(60-t)^2 \\ \tfrac{(60)^2}{3} &= (60-t)^2 \\ \tfrac{60}{\sqrt{3}} &= 60-t \\ t &= 60 - \tfrac{60}{\sqrt{3}} \end{aligned}$$

The maximum amount of salt in the tank is

$$X(60 - \tfrac{60}{\sqrt{3}}) = \tfrac{60}{\sqrt{3}} - \tfrac{1}{(60)^2}(\tfrac{60}{\sqrt{3}})^3 = \tfrac{60}{\sqrt{3}} - \tfrac{60}{3\sqrt{3}} = \tfrac{60}{\sqrt{3}}(\tfrac{2}{3}) = \boxed{\tfrac{40}{\sqrt{3}}\text{lbs.}}$$

7. Section 1.6: Substitution Techniques

In this section we solve three types of problems.

(1) (Homogeneous substitution) To solve $\frac{dy}{dx} = F(\frac{y}{x})$, let $v = \frac{y}{x}$, turn the Differential Equation into a DE which involves the function $v = v(x)$. You will be able to separate the variables.
(2) (Linear substitution) To solve $\frac{dy}{dx} = F(ax+by+x)$, let $v = ax+by+c$, turn the Differential Equation into a DE which involves the function $v = v(x)$. You will be able to separate the variables.
(3) (Bernoulli equation) To solve $y' + P(x)y = y^n Q(x)$, let $v = y^{1-n}$, turn the Differential Equation into a DE which involves the function $v = v(x)$. The new problem will be a First Order Linear problem.

Example. Solve $x^2y' + 2xy = 5y^4$.

Answer: If need be, divide both sides by x^2 to obtain

$$y' + \tfrac{2}{x}y = \tfrac{5}{x^2}y^4. \tag{7.0.1}$$

This is a Bernoulli Equation $y' + P(x)y = y^n Q(x)$ with $P(x) = \frac{2}{x}$, $Q(x) = \frac{5}{x^2}$, and $n = 4$. Let $v = y^{1-n} = y^{-3}$. Calculate $\frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$. Multiply both sides of (7.0.1) by $-3y^{-4}$ to obtain

$$-3y^{-4}y' + \frac{2}{x}(-3y^{-4})y = \frac{5}{x^2}(-3y^{-4})y^4$$
$$\tfrac{dv}{dx} + (\tfrac{-6}{x})v = \tfrac{-15}{x^2}$$

which is a first order linear DE as we expected! Multiply both sides by

$$\mu(x) = e^{\int \frac{-6}{x}dx} = e^{-6\ln x} = x^{-6}$$

to obtain

$$x^{-6}\tfrac{dv}{dx} - 6x^{-7}v = -15x^{-8}.$$

Observe that the left side is the derivative of $x^{-6}v$ with respect to x. Integrate both sides with respect to x to obtain

$$x^{-6}v = \frac{15}{7}x^{-7} + C$$

or

$$v = \frac{15}{7}x^{-1} + Cx^6$$

or

$$y^{-3} = \frac{15}{7}x^{-1} + Cx^6.$$

Of course, we must solve for y:

$$\boxed{\frac{1}{(\frac{15}{7}x^{-1} + Cx^6)^{1/3}} = y.}$$

Check: Plug the proposed answer into the left side of the DE to obtain:

$$x^2y'+2xy=x^2\left(-\frac{1}{3}\right)\left(\frac{15}{7}x^{-1}+Cx^6\right)^{-4/3}\left(-\frac{15}{7}x^{-2}+6Cx^5\right)+2x\left(\frac{15}{7}x^{-1}+Cx^6\right)^{-1/3}.$$

Factor out the lowest power of $(\frac{15}{7}x^{-1}+Cx^6)$ that appears to see that

$$\begin{aligned}x^2y'+2xy&=\left(\frac{15}{7}x^{-1}+Cx^6\right)^{-4/3}\left[x^2\left(-\frac{1}{3}\right)\left(-\frac{15}{7}x^{-2}+6Cx^5\right)+2x\left(\frac{15}{7}x^{-1}+Cx^6\right)\right]\\&=\left(\frac{15}{7}x^{-1}+Cx^6\right)^{-4/3}\left[+\frac{5}{7}-2Cx^6+\frac{30}{7}+2Cx^6\right]=5y^4\checkmark\end{aligned}$$

Example. Solve

$$y'=\sqrt{x+y+1}. \tag{7.0.2}$$

This problem has the form $y'=f(L)$ for some expression that is linear in both x and y. We make a **linear substitution**. We let $v=x+y+1$. We turn the DE into a DE that involves $v=v(x)$. We are guaranteed that we will be able to separate the variables.

Observe that $\frac{dv}{dx}=1+\frac{dy}{dx}$. The original problem becomes

$$\frac{dv}{dx}-1=\sqrt{v}$$

or

$$\frac{dv}{dx}=\sqrt{v}+1$$

or

$$\frac{dv}{\sqrt{v}+1}=dx.$$

Integrate both sides

$$\int\frac{dv}{\sqrt{v}+1}=\int dx. \tag{7.0.3}$$

To integrate $\int\frac{dv}{\sqrt{v}+1}$, we make a sneaky substitution. Let $u=\sqrt{v}$. It follows that $\frac{du}{dv}=\frac{1}{2\sqrt{v}}$. We may rewrite the most recent equation as $2udu=dv$. Thus,

$$\begin{aligned}\int\frac{dv}{\sqrt{v}+1}&=\int\frac{2udu}{u+1}=\int\frac{2(u+1)-2}{u+1}du=\int\left(2-\frac{2}{u+1}\right)du\\&=2u-2\ln(u+1)+C=2\sqrt{v}-2\ln(\sqrt{v}+1)+C.\end{aligned}$$

By the way, it is easy to check that

$$\frac{d}{dv}\left(2\sqrt{v}-2\ln(\sqrt{v}+1)\right)=\frac{1}{\sqrt{v}+1}.$$

At any rate, we see that (7.0.3) yields

$$2\sqrt{v}-2\ln(\sqrt{v}+1)+C=x,$$

or

$$2\sqrt{x+y+1}-2\ln(\sqrt{x+y+1}+1)+C=x.$$

I would like to solve the most recent equation for y. If I could, I would. But I can't. The best I can do is write,

> Any function $y = y(x)$ which satisfies $2\sqrt{x+y+1} - 2\ln(\sqrt{x+y+1}+1) + C = x$ is a solution of the differential equation $\frac{dy}{dx} = \sqrt{x+y+1}$.

Check. So how does one check the above answer? Well, one can use implicit differentiation to find $\frac{dy}{dx}$. Then verify that $\frac{dy}{dx}$ is equal to $\sqrt{x+y+1}$.

Suppose y is a function of x and y satisfies

$$2\sqrt{x+y+1} - 2\ln(\sqrt{x+y+1}+1) + C = x, \tag{7.0.4}$$

then we can use implicit differentiation to find $\frac{dy}{dx}$. One merely takes $\frac{d}{dx}$ of both sides; every time one needs to find $\frac{d}{dx}$ of y, one merely writes $\frac{dy}{dx}$. Eventually one has an equation which involves x, y, and $\frac{dy}{dx}$. One solves for $\frac{dy}{dx}$. Take $\frac{d}{dx}$ of both sides of (7.0.4):

$$2\left(\frac{1+\frac{dy}{dx}}{2\sqrt{x+y+1}}\right) - 2\frac{\frac{1+\frac{dy}{dx}}{2\sqrt{x+y+1}}}{\sqrt{x+y+1}+1} = 1.$$

Now we solve for $\frac{dy}{dx}$. Cancel the two 2's in the first summand; cancel the two 2's in the second summand; and multiply the top and the bottom of the second summand by $\sqrt{x+y+1}$ in order to obtain

$$\left(\frac{1+\frac{dy}{dx}}{\sqrt{x+y+1}}\right) - \frac{1+\frac{dy}{dx}}{\sqrt{x+y+1}(\sqrt{x+y+1}+1)} = 1.$$

Multiply both sides by $\sqrt{x+y+1}(\sqrt{x+y+1}+1)$ to obtain:

$$\left(1+\frac{dy}{dx}\right)(\sqrt{x+y+1}+1) - \left(1+\frac{dy}{dx}\right) = \sqrt{x+y+1}(\sqrt{x+y+1}+1).$$

Factor the left side and get

$$\left(1+\frac{dy}{dx}\right)(\sqrt{x+y+1}+1-1) = \sqrt{x+y+1}(\sqrt{x+y+1}+1)$$

or

$$\left(1+\frac{dy}{dx}\right)\sqrt{x+y+1} = \sqrt{x+y+1}(\sqrt{x+y+1}+1).$$

If $\sqrt{x+y+1}$ is non-zero, then divide both sides by $\sqrt{x+y+1}$.

$$1+\frac{dy}{dx} = \sqrt{x+y+1}+1.$$

Subtract 1 from each side to see that

$$\frac{dy}{dx} = \sqrt{x+y+1}.$$

Thus, any function $y = y(x)$ which satisfies (7.0.4) is a solution of the Differential Equation (7.0.2).[7]

Example. Solve

$$2xyy' = x^2 + 2y^2.$$

Notice that the DE is not linear; one can not separate the variables; and one can not make a linear substitution. One could treat it as a Bernoulli equation. I will make a homogeneous substitution. Recall that if one is given a DE of the form $\frac{dy}{dx} = f(\frac{y}{x})$, then one can let $v = \frac{y}{x}$ and turn the DE into an equation involving $y(v)$. One is guaranteed that one can separate the variables in the resulting equation. Maybe you don't see any $\frac{y}{x}$'s in the given DE. Think about the meaning of the word "homogeneous". It means well-mixed. When we buy milk now, it is always homogenized: somebody shook it up so that it has the same consistency throughout the whole container. When my father was young, milk was not homogenized: the cream was on the top and other parts layered below. The given DE is homogeneous because every terms has the same degree (namely 2) in the symbols $\{x, y\}$. If we divide by x^2, then we will end up with y' is equal to some function of $\frac{y}{x}$. There will not be any left over unattached x's or y's.

Divide the given DE by x^2 to obtain

$$2\left(\frac{y}{x}\right)y' = 1 + 2\left(\frac{y}{x}\right)^2.$$

Let $v = \frac{y}{x}$. It follows that $xv = y$. Use the product rule to see that

$$x\frac{dv}{dx} + v = \frac{dy}{dx}.$$

The DE has become

$$2v(x\frac{dv}{dx} + v) = 1 + 2v^2$$

or

$$x\frac{dv}{dx} + v = \frac{1}{2v} + v$$

or

$$x\frac{dv}{dx} = \frac{1}{2v}.$$

Separate the variables

$$v\,dv = \frac{1}{2x}dx.$$

Integrate both sides

$$\frac{v^2}{2} = \frac{1}{2}\ln|x| + C.$$

[7] I want to emphasize that whenever possible one solves a DE of the form $\frac{dy}{dx} = f(x, y)$ for y. But if one is not able to solve for y, then one leaves the answer in the form where $y(x)$ is implicitly given by some equation which involves x and y like (7.0.4). Observe that (7.0.4) is not as good as $y = y(x)$; but it is MUCH better than (7.0.2). In particular, (7.0.4) does not involve and derivatives.

Multiply both sides by 2 and take the square root of both sides

$$v = \pm\sqrt{\ln|x| + 2C}.$$

Of course $v = \frac{y}{x}$. Multiply both sides by x and rename $2C$ to be K to obtain

$$\boxed{y = \pm x\sqrt{\ln|x| + K}.}$$

Check. We assume that x is positive and we check $y = x\sqrt{\ln x + K}$. Plug

$$y = x\sqrt{\ln x + K}$$

into the left side of the original DE:

$$\begin{aligned} 2xyy' &= 2x^2\sqrt{\ln x + K}\left(x\frac{\frac{1}{x}}{2\sqrt{\ln x + K}} + \sqrt{\ln x + K}\right) \\ &= x^2 + 2x^2(\sqrt{\ln x + K})^2 \\ &= x^2 + 2y^2. \checkmark \end{aligned}$$

8. Section 2.1: Population Models

Let $P(t)$ be the size of some population at time t. In this section we study two DE:

(a) $\frac{dP}{dt} = kP(M-P)$
(b) $\frac{dP}{dt} = kP(P-M)$

In each DE k and M are positive constants and k is very small. Equation (a) is called the logistic equation. It is often used to model population growth. The mathematics for solving Equation (b) is the same as the mathematics used to solve (a) but the result is much different. Equation (b) is called the explosion/extinction equation.

8.A. **Why the logistic equation is a good model for population growth.** Lets examine (a) more carefully:

$$\frac{dP}{dt} = kMP - kP^2$$

with k very small. When P is small, then the $-kP^2$ term does not matter much and the DE is almost $\frac{dP}{dt} = kMP$ which is the DE for exponential growth. Ah, but when P is large, then the P^2 factor overcomes the small k and the $-kP^2$ term becomes significantly more important. *As the population becomes large, there are forces (scarcity of food or shelter, or abundance of disease) which keep the population in check.* I hope it is clear that the logistic equation is a better model for population growth than merely DE $\frac{dP}{dt} = (\text{constant})P$.

8.B. **We solve the logistic equation.** We solve

$$\frac{dP}{dt} = kMP - kP^2.$$

We can separate the variables:

$$\frac{dP}{MP-P^2} = k\,dt. \tag{8.0.1}$$

It is not difficult to integrate

$$\frac{1}{P(M-P)}.$$

We use the technique of partial fractions and see what $\frac{1}{P(M-P)}$ used to look like before some body "cleaned it up". It used to be

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

for some numbers A and B. We can figure out A and B. Clear the denominators:

$$1 = A(M-P) + BP.$$

(The last equation holds for all P. The number M is fixed and not zero. Our job is to find A and B.) Plug in $P=M$ to learn that $\frac{1}{M}=B$. Plug in $P=0$ to learn that $\frac{1}{M}=A$. Observe that

$$\frac{1}{M}\left(\frac{1}{P}+\frac{1}{M-P}\right)$$

really does equal $\frac{1}{P(M-P)}$. Integrate Equation (8.0.1) to obtain

$$\frac{1}{M}\int\left(\frac{1}{P}+\frac{1}{M-P}\right)dP=\int kdt$$

and

$$\frac{1}{M}(\ln P-\ln|M-P|)=kt+C.$$

(There is no need to write $|P|$, because P is a population; hence it can not be negative.) Multiply both sides of the equation by M to get

$$\ln\left(\frac{P}{|M-P|}\right)=Mkt+MC$$

Exponentiate to obtain

$$\frac{P}{|M-P|}=e^{MC}e^{Mkt}.$$

Of course, $|M-P|=\pm(M-P)$. Move $\pm$ to the other side and let $K=\pm e^{MC}$.

$$\frac{P}{M-P}=Ke^{Mkt}. \tag{8.0.2}$$

This is a good time to calculate K. Plug $t=0$ into both sides of (8.0.2) to learn that

$$\frac{P(0)}{M-P(0)}=K. \tag{8.0.3}$$

We want to solve for P; so we multiply both sides of (8.0.2) by $M-P$ to obtain

$$P=Ke^{Mkt}(M-P).$$

Add $Ke^{Mkt}P$ to both sides

$$P(1+Ke^{Mkt})=Ke^{Mkt}M.$$

Divide both sides by $1+Ke^{Mkt}$ and obtain

$$P=\frac{Ke^{Mkt}M}{1+Ke^{Mkt}}.$$

This is **a** formula for $P(t)$; but lets clean it up a little! Instead of having two competing exponential functions, we arrange things so that there is only one exponential function. Instead of having the constant K appear twice, we arrange things so K only appears once. Divide top and bottom by Ke^{Mkt}. Thus,

$$P=\frac{M}{\frac{e^{-Mkt}}{K}+1}.$$

Replace K by the value given in (8.0.3).

$$P = \frac{M}{\frac{(M-P(0))e^{-Mkt}}{P(0)} + 1}.$$

Multiply top and bottom by $P(0)$. We have calculated that

$$P(t) = \frac{MP(0)}{(M-P(0))e^{-Mkt} + P(0)},$$

or

$$P(t) = \frac{MP(0)}{P(0) + (M-P(0))e^{-Mkt}}.$$

Here is the first cool observation. If the population $P(t)$ is governed by the logistic equation, then the population is sustainable. That is,

$$\lim_{t\to\infty} P(t)$$

exists and is finite. In particular,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP(0)}{P(0) + (M-P(0))e^{-Mkt}} = \frac{MP(0)}{P(0)} = M,$$

because $\lim\limits_{t\to\infty} e^{-Mkt} = 0$.

Here is the second observation. If the population $P(t)$ starts smaller than the limiting population (i.e. M), then the population will always be smaller than M because the denominator of

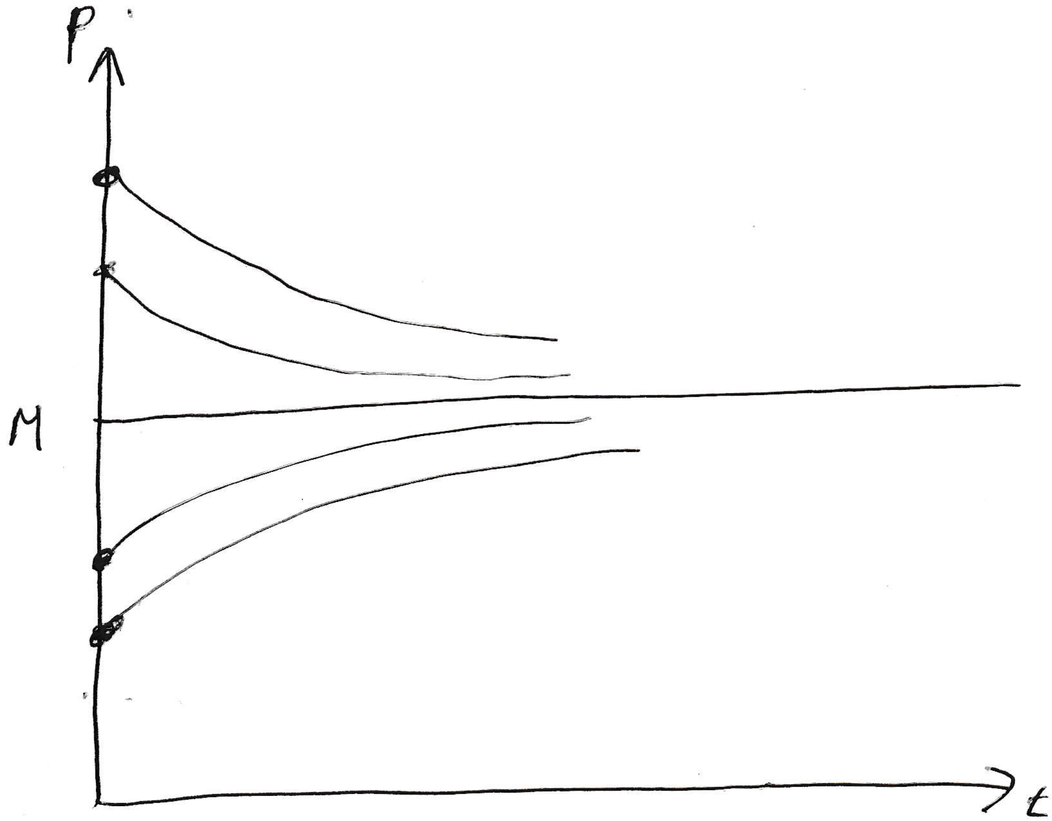
$$P(t) = \frac{MP(0)}{P(0) + (M-P(0))e^{-Mkt}}$$

will always be bigger than $P(0)$; hence the ratio $\frac{P(0)}{P(0)+(M-P(0))e^{-Mkt}}$ will always be LESS THAN 1; so $P(t)$ is some fraction times M.

Similarly, if the population starts larger than the limiting population, then the population will always be larger than M because the multiplier $\frac{P(0)}{P(0)+(M-P(0))e^{-Mkt}}$ will always be larger than one.

The graph of $P(t)$ for various choices of $P(0)$ looks like:

Solutions of the logistic equation



$$\frac{dP}{dt} = kP(M-P) \qquad \text{with } 0 < k, M$$

Example 8.1. We do problem 15 from section 2.1. Consider a population $P(t)$ which satisfies the logistic equation $\frac{dP}{dt} = aP - bP^2$, where a and b are constants, $B = aP$ is the birth rate, and $D = bP^2$ is the death rate. Write M in terms of $B(0)$, $D(0)$, and $P(0)$.

The point of the problem is that we know the solution of $\frac{dP}{dt} = kP(M-P)$. Indeed, we know that in the long term the population will approach M.

Someone interested in the population can probably calculate $B(0)$, $D(0)$, and $P(0)$. If we can express M in terms of $B(0)$, $D(0)$, and $P(0)$, then we can make a plausible prediction of what the population will do without doing anymore calculating.

Look at the two equations:

$$\frac{dP}{dt} = kMP - kP^2$$
$$\frac{dP}{dt} = aP - bP^2$$

The coefficient of P^2 is $-k$ in the top equation and is $-b$ in the bottom equation; so, $k = b$. The coefficient of P is kM in the top equation and is a in the bottom equation. Thus $a = kM$; so $a = bM$ and $\frac{a}{b} = M$.

Plug 0 into $B = aP$ and $D = aP^2$ to learn that $B(0) = aP(0)$ and $D(0) = bP(0)^2$. We conclude that $M = \frac{a}{b} = \frac{\frac{B(0)}{P(0)}}{\frac{D(0)}{P(0)^2}} = \frac{B(0)P(0)}{D(0)}$. Our answer is

$$\boxed{M = \frac{B(0)P(0)}{D(0)}.}$$

Example 8.2. We do problem 17 from section 2.1. **Here is the statement of the problem: The Logistic Equation is $\frac{dP}{dt} = kP(M-P)$, where k and M are positive constants. The solution of the Logistic Equation is**

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{-kMt}}.$$

Recall that if a population $P(t)$ satisfies the logistic equation

$$\frac{dP}{dt} = aP - bP^2,$$

where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur, then the limiting population is

$$M = \lim_{t\to\infty} P(t) = \frac{B(0)P(0)}{D(0)}.$$

Consider a rabbit population $P(t)$ which satisfies the logistic equation. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

Here is the solution of the problem: We are told that $P(0) = 240$, $B(0) = 9$, $D(0) = 12$. We calculate

$$M = \frac{B(0)P(0)}{D(0)} = \frac{9(240)}{12} = 180$$

and

$$k = b = \frac{D(0)}{P(0)^2} = \frac{12}{(240)^2} = \frac{1}{(240)(20)}.$$

We must find t so that

$$\frac{105}{100}(180) = \frac{180(240)}{240 + (180-240)e^{-180t/((240)20)}}.$$

Cancel 180 from the left and the right. Divide top and bottom on the right by 60.

$$\frac{105}{100} = \frac{4}{4 - e^{-(3/80)t}}$$

$$4 - e^{-(3/80)t} = 4\left(\frac{100}{105}\right)$$

$$4 - 4\left(\frac{100}{105}\right) = e^{-(3/80)t}$$

$$4\left(\frac{5}{105}\right) = e^{-(3/80)t}$$

$$\ln\left(\frac{20}{105}\right) = -(3/80)t$$

$$\ln\left(\frac{105}{20}\right) = (3/80)t$$

$$\boxed{\frac{80}{3}\ln\left(\frac{105}{20}\right) \text{ months} = t}.$$

Problem 18 in Section 2.1 Consider a population $P(t)$ which satisfies the extinction/explosion Differential Equation $\frac{dP}{dt} = aP^2 - bP$, where $B = aP^2$ is the time rate at which births occur and $D = bP$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$ and B_0 births per month and D_0 births per month are occurring at time $t = 0$, show that the threshold population is $M = D_0P_0/B_0$.

Solution. Compare the two forms of the extinction-explosion Differential Equation:

$$\frac{dP}{dt} = kP^2 - kMP$$
$$\frac{dP}{dt} = aP^2 - bP$$

to see that $a = k$ and $b = kM$. Plug $t = 0$ into the equations $B = aP^2$ and $D = bP$ to see that $B(0) = aP(0)^2$ and $D(0) = bP(0)$. Conclude that

$$M = \frac{b}{k} = \frac{b}{a} = \frac{\frac{D(0)}{P(0)}}{\frac{B(0)}{P(0)^2}} = \frac{D(0)}{P(0)}\frac{P(0)^2}{B(0)} = \frac{D(0)P(0)}{B(0)} = \frac{D_0P_0}{B_0}.$$

Problem 19 in Section 2.1 Consider an alligator population which satisfies the extinction/explosion Differential Equation as in Problem 18. If the initial population is 100 alligators and there are 10 births per month and 9 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 10 times the threshold population M?

Solution We saw in number 18 that $M = \frac{D(0)P(0)}{B(0)}$. This problem has

$$P(0) = 100, \quad B(0) = 10, \quad \text{and} \quad D(0) = 9.$$

Thus $M = \frac{9(100)}{10} = 90$. We want to find t with $P(t) = (90)(10) = 900$. The solution of the Initial Value Problem

$$\frac{dP}{dt} = kP(P-M) \quad P(0) = P_0 \quad \text{with } k \text{ and } M \text{ positive}$$

is

$$P = \frac{MP_0}{P_0 + (M-P_0)e^{kMt}}.$$

(See problem 33 or the class notes. There is no reason to memorize this formula.) Of course the Differential Equations

$$\frac{dP}{dt} = kP^2 - kMP$$
$$\frac{dP}{dt} = aP^2 - bP$$

are exactly the same if one takes $a = k$ and $b = kM$, where $B = aP^2$ and $D = bP$. In particular, $k = a = \frac{B(0)}{P(0)^2} = \frac{B_0}{P_0^2} = \frac{10}{100^2} = \frac{1}{1000}$. Thus,

$$P = \frac{MP_0}{P_0 + (M-P_0)e^{kMt}} = \frac{90(100)}{100 + (-10)e^{\frac{90}{1000}t}}.$$

Our job is to find t with

$$900 = \frac{90(100)}{100 + (-10)e^{\frac{90}{1000}t}}.$$

Multiply both sides by $100 - 10e^{9t/100}$. Divide both sides by 900.

$$100 - 10e^{\frac{9}{100}t} = 10.$$

Subtract 10 from both sides. Add $10e^{\frac{9}{100}t}$ to both sides. Obtain

$$90 = 10e^{\frac{9}{100}t}.$$

$$9 = e^{\frac{9}{100}t}$$
$$\ln 9 = \frac{9}{100}t$$
$$\frac{100}{9}\ln 9 = t.$$

$$\boxed{\text{The population will reach 900, which is } 10M\text{, after } \frac{100}{9}\ln 9 \text{ months.}}$$

8.C. **We solve the explosion/extinction equation.** We solve
$$\frac{dP}{dt} = kP(P-M),$$
with k and M positive. We separate the variables and do the partial fractions:
$$\frac{dP}{P(P-M)} = kdt,$$
$$\int \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right) dP = \int kdt,$$
$$\ln|P-M| - \ln P = Mkt + C,$$
$$\frac{|P-M|}{P} = e^C e^{Mkt},$$
$$\frac{P-M}{P} = Ke^{Mkt},$$
where $K = \pm e^C$,
$$P - M = Ke^{Mkt}P.$$
At this point we calculate that $\frac{P(0)-M}{P(0)} = K$. Move all the terms with P to the left and all of the terms without P to the right:
$$P(1 - Ke^{Mkt}) = M,$$
$$P(t) = \frac{M}{1 - Ke^{Mkt}},$$
$$P(t) = \frac{M}{1 - \frac{(P(0)-M)}{P(0)}e^{Mkt}},$$
$$P(t) = \frac{P(0)M}{P(0) - (P(0)-M)e^{Mkt}}.$$
We draw some conclusions.

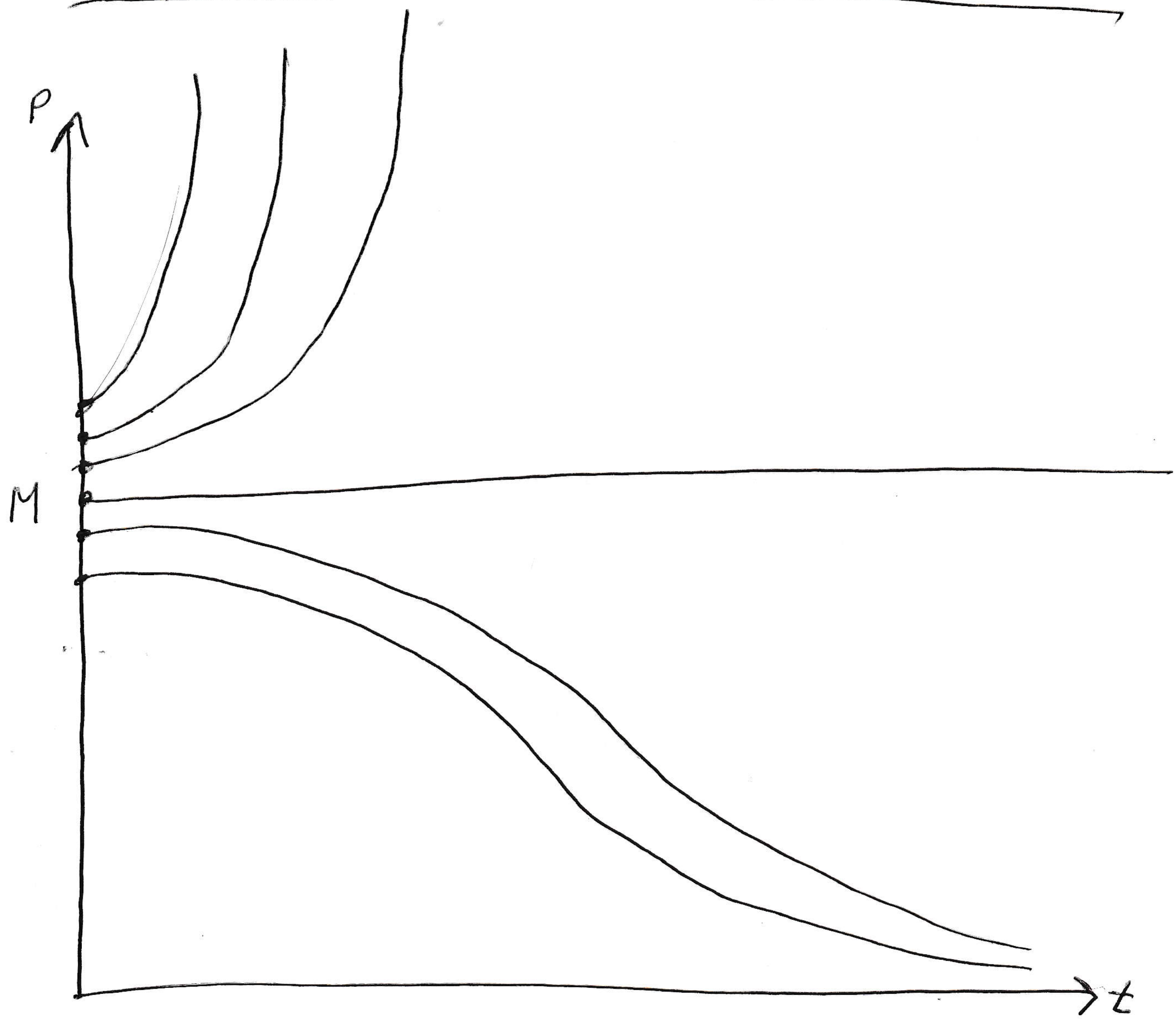
- If $P(0) = M$, then $P(t) = M$ for all t.
- If $P(0) < M$, then the denominator is always positive and goes to $+\infty$ as t goes to ∞. Thus, in this case $\lim_{t\to\infty} P(t) = 0$. (In this case the population becomes extinct!)

- If $M < P(0)$, then the denominator starts positive but eventually goes to $-\infty$ Thus, there is a finite time when the denominator becomes 0. In other words, there is a finite time when the population $P(t)$ **explodes** to $+\infty$. (In problem 45 in section 1.1, see also Example 5.1 in these notes, a population of rats exploded.)

The DE $\frac{dP}{dt} = kP(P-M)$ is called the extinction/explosion DE because there is a magic threshold. If the initial population is less than the threshold, then the population will die out. If the initial population is above the threshold, then the population will grow out of control. Neither of these situations is sustainable!

The graph of $P(t)$ for various choices of $P(0)$ looks like:

Solutions of the Extinction/Explosion Equation



$$\frac{dP}{dt} = kP(P-M) \qquad \text{with } 0 < k, M$$

9. SECTION 2.2: EQUILIBRIUM AND STABILITY.

This section is about Differential Equations of the form $\frac{dy}{dx} = f(y)$. The authors call such Differential Equations autonomous.

When a constant function $y = c$ is a solution of an autonomous DE equation, then this solution is called an equilibrium solution of the DE.

If $y = c$ is an equilibrium solution of a DE and all solutions of the DE which come near $y = c$ are tangent to $y = c$, then $y = c$ is stable equilibrium.

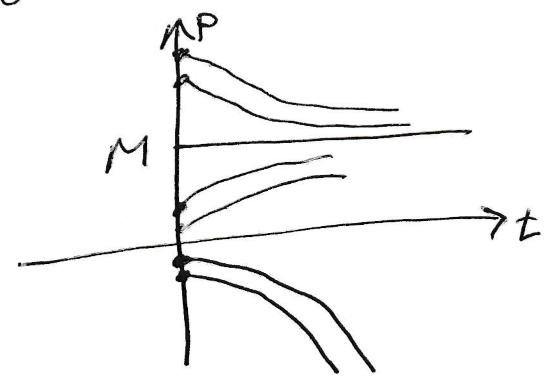
If $y = c$ is an equlibrium solution of a DE, but $y = c$ is not a stable equilibrium, then $y = c$ is an unstable equilibrium.

Examples of Equilibrium

$P = M$ is a stable equilibrium solution of the logistic equation

$$\frac{dP}{dt} = kP(M-P)$$ where k, M are positive constants

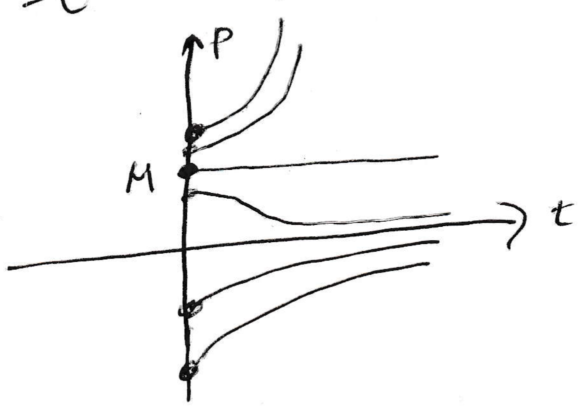
$P = 0$ is an unstable equilibrium solution of the logistic equation



$P = M$ is an unstable equilibrium solution of the EE equation

$$\frac{dP}{dt} = kP(P-M)$$ where k, M are positive constants

$P = 0$ is a stable equilibrium solution of the EE equation



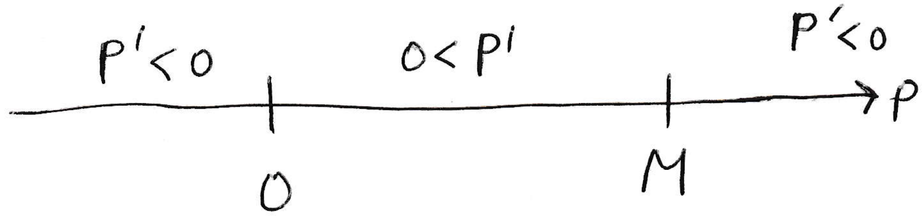
Of course, if c is a number with $f(c) = 0$, then $y = c$ is an equilibrium solution of the DE $\frac{dy}{dx} = f(y)$ (because $\frac{d}{dx}(c) = 0$.

Once one has identified the equilibrium solutions of $\frac{dy}{dx} = f(y)$, then one can look at the "phase diagram" to see if the equilibrium solution is stable. Draw the y number line and mark the equilbrium solutions. The y number line has been chopped into a handful of intervals and y' has constant sign on each of these intevals. So y is always increasing or always decreasing on each of these intervals. Without making any calculation, you can see if an arbitrary solution of the DE is heading toward or away from the nearest equilibrium solution.

Phase Diagrams

The phase diagram for the logistic equation $\frac{dP}{dt} = kP(M-P)$

is



Every solution that has $M < P$ is decreasing down toward $P = M$

Every solution that has $0 < P < M$ is increasing. This solution is fleeing the equilibrium solution $P = 0$ but is racing toward the equilibrium solution $P = M$.

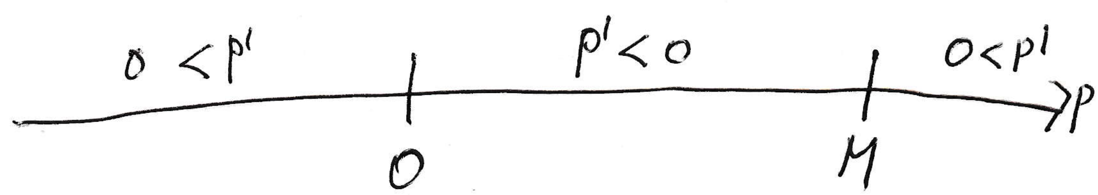
Every solution that has $P < 0$ is decreasing away from the equilibrium solution $P = 0$.

The phase diagram alone tells us that

$P = M$ is a stable Equilibrium

$P = 0$ is an unstable Equilibrium.

The phase diagram for the EE-equation $\frac{dP}{dt} = kP(P-M)$



Again the diagram alone shows that $P = M$ is an unstable equilibrium, but $P = 0$ is a stable equilibrium

10. Sections 3.1 and 3.2: Higher Order Linear Differential Equations.

Definition.

(a) A Differential Equation of the form

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = Q(x) \tag{10.0.1}$$

is called an n^{th}-order linear Differential Equation.

(b) If $Q(x) = 0$, then the Differential Equation (10.0.1) is called an n^{th}-order homogeneous linear Differential Equation.

Remarks.

(a) The Differential Equation (10.0.1) is **linear** in the symbols $y, y', \dots, y^{(n)}$. Each of these symbols appears with power zero or power one. No terms involves more than one of these symbols.

(b) A homogeneous linear Differential Equation is homogeneous in the sense that every term has degree exactly **one** in the symbols $y, y', \dots, y^{(n)}$.

(c) The appropriate Initial Condition for (10.0.1) is

$$y(a) = b_0, \quad y'(a) = b_1, \quad \dots, \quad y^{(n-1)}(a) = b_{n-1} \tag{10.0.2}$$

where $a, b_0, \dots, b_{n-1}$ all are numbers. Notice that there are n-parts to the Initial Condition for an n^{th} order Differential Equation.

Theorem. The existence and uniqueness theorem for Higher Order Linear Differential Equations. *Consider the Initial Value problem*

$$\begin{cases} y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = Q(x), & \text{(DE)} \\ y(a) = b_0, \quad y'(a) = b_1, \quad \dots, \quad y^{(n-1)}(a) = b_{n-1} & \text{(IC)} \end{cases} \tag{10.0.3}$$

If $P_1(x), \dots, P_n(x)$ and $Q(x)$ all are continuous on some open interval I which contains a, then the Initial Value Problem (10.0.3) *has a unique solution $y = y(x)$ which is defined on all of I.*

Here are the steps for solving the Initial Value Problem (10.0.3).

Step 1. Find n linearly independent [8] solutions $y_1, \dots, y_n$ of the homogeneous problem[9]

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0.$$

[8] "linearly Independent" is a subtle concept. It is a sophisticated way of saying "really really different". If you are expecting two different solutions, you would not be satisfied if I gave you the same solution twice. You also would not be satisfied if I gave you a solution and then a constant times the first solution. Similarly, if you were expecting three really different solutions, you would not be satisfied if I gave you two different solutions and then the sum of the first two solutions as the third solutions. The functions $y_1, \dots, y_n$ are linearly independent if it is impossible to write any of the functions y_i in terms of the rest of the functions with constant coefficients.

[9] The linear homogeneous Differential Equation has a magical property. If $Y_1, \dots, Y_N$ are solutions of a linear homogeneous Differential Equation, then $c_1Y_1 + \cdots + c_NY_N$ is also a solution of the Differential Equation for all constants c_i.

Step 2. Find one particular solution y_{partic} of the original Differential equation (DE) from (10.0.3).

At this point $y = c_1y_1 + \cdots + c_ny_n + y_{\text{partic}}$ is the general solution of the original Differential Equation (DE) from (10.0.3).

Step 3. Find $c_1, \ldots, c_n$ so that the Initial Condition (IC) from (10.0.3) is satisfied.

Example. Consider the Initial Value Problem

$$y'' + y = e^x \tag{10.0.4}$$

$$y(0) = -\frac{1}{2}, \quad y'(0) = \frac{3}{2} \tag{10.0.5}$$

(a) Verify that $y = \sin x$ and $y = \cos x$ are solutions of the homogeneous problem $y'' + y = 0$.
(b) Verify that $y = \frac{1}{2}e^x$ is a solution of (10.0.4).
(c) Verify that $y = c_1 \sin x + c_2 \cos x + \frac{1}{2}e^x$ is a solution of (10.0.4) for all constants c_1 and c_2.
(d) Solve (10.0.4) and (10.0.5).

We get to work.

(a) We plug $y = \sin x$ into the left side $y'' + y = 0$. If $y = \sin x$, then $y' = \cos x$ and $y'' = -\sin x$; thus,

$$y'' + y = -\sin x + \sin x = 0. \checkmark.$$

We plug $y = \cos x$ into the left side $y'' + y = 0$. If $y = \cos x$, then $y' = -\sin x$ and $y'' = -\cos x$; thus,

$$y'' + y = \cos x - \cos x = 0. \checkmark.$$

(b) We plug $y = \frac{1}{2}e^x$ into the left side $y'' + y = e^x$. If $y = \frac{1}{2}e^x$, then $y' = \frac{1}{2}e^x$ and $y'' = \frac{1}{2}e^x$; thus,

$$y'' + y = \frac{1}{2}e^x + \frac{1}{2}e^x = e^x. \checkmark$$

(c) We plug $y = c_1 \sin x + c_2 \cos x + \frac{1}{2}e^x$ into the left side of $y'' + y = e^x$. If

$$y = c_1 \sin x + c_2 \cos x + \frac{1}{2}e^x,$$

then

$$y' = c_1 \cos x - c_2 \sin x + \frac{1}{2}e^x;$$

$$y'' = -c_1 \sin x - c_2 \cos x + \frac{1}{2}e^x;$$

and

$$y'' + y = \left(-c_1 \sin x - c_2 \cos x + \frac{1}{2}e^x\right) + \left(c_1 \sin x + c_2 \cos x + \frac{1}{2}e^x\right) = e^x. \checkmark$$

(d) We find c_1 and c_2 so that $y = c_1 \sin x + c_2 \cos x + \frac{1}{2}e^x$ has the property that

$$y(0) = -\frac{1}{2} \quad \text{and} \quad y'(0) = \frac{3}{2}.$$

We know $y = c_1 \sin x + c_2 \cos x + \frac{1}{2}e^x$; so $y(0) = c_2 + \frac{1}{2}$. We know

$$y' = c_1 \cos x - c_2 \sin x + \frac{1}{2}e^x;$$

so $y'(0) = c_1 + \frac{1}{2}$. We must solve the system of equations

$$-\frac{1}{2} = c_2 + \frac{1}{2} \quad \text{and} \quad \frac{3}{2} = c_1 + \frac{1}{2}.$$

Thus,

$$-1 = c_2 \quad \text{and} \quad 1 = c_1.$$

The solution of the Initial Value Problem (10.0.4) and (10.0.5) is

$$\boxed{y = \sin x - \cos x + \frac{1}{2}e^x.}$$

11. Section 3.3: Linear Homogeneous Differential Equations with constant coefficients.

In this section we solve

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where the a_i's are constants.

Example. Solve

$$y'' - 3y' + 2y = 0. \tag{11.0.1}$$

Try $y = e^{rx}$, where r is a constant. Compute $y' = re^{rx}$ and $y'' = r^2e^{rx}$. If $y = e^{rx}$ is a solution of (11.0.1), then

$$r^2e^{rx} - 3rr^{rx} + 2e^{rx} = 0.$$

Thus,

$$e^{rx}(r^2 - 3r + 2) = 0.$$

If a product of two numbers is zero, then one of the numbers is zero. The function e^{rx} is never zero; hence $r^2 - 3r + 2 = 0$; $(r-2)(r-1) = 0$; and $r = 2$ or $r = 1$. We easily check that $y = e^{2x}$ and $y = e^x$ both are solutions of (11.0.1). (Indeed. if $y = e^2x$, then $y' = 2e^{2x}$, $y'' = 4e^{2x}$, and $y'' - 3y' + 2y = 4e^{2x} - 3(2e^{2x}) + 2e^{2x} = e^{2x}(4 - 6 + 2) = 0$. If $y = e^x$, then $y' = e^x$, $y'' = e^x$, and $y'' - 3y' + 2y = e^x - 3(e^x) + 2e^x = e^x(1 - 3 + 2) = 0$.) A particular solution of (11.0.1) is $y_{partic} = 0$. Thus, the general solution of (11.0.1) is

$$y = c_1e^{2x} + c_2e^x,$$

where c_1 and c_2 are arbitrary constants.

To solve

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{11.0.2}$$

where the a_i's are constants:

Try $y = e^{rx}$. Study the characteristic equation

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{11.0.3}$$

(a) If (11.0.3) has n distinct real roots $r_1, \dots, r_n$, then the general solution of (11.0.2) is

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx}.$$

(b) Special care must be taken if the characteristic equation (11.0.3) has repeated roots.

(c) Special care must be taken if the characteristic equation (11.0.3) has non-real roots.

11.A. **Suppose the characteristic equation has repeated real roots.**

Fact 11.1. *Suppose you are trying to solve*

$$(11.0.2) \qquad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

and the corresponding characteristic equation

$$(11.0.3) \qquad a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

has a factor of the from $(r - r_1)^m$*, then the functions*

$$y_1 = e^{r_1 x},\ y_2 = xe^{r_1 x},\ y_3 = x^2 e^{r_1 x},\ \ldots,\ y_m = x^{m-1} e^{r_1 x}$$

all are solutions of (11.0.2).

Example. Solve

$$(11.1.1) \qquad y'' - 2y' + y = 0.$$

We try $y = e^{rx}$. We compute $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. If $y = e^{rx}$ is a solution of (11.1.1), then[10] $e^{rx}(r^2 - 2r + 1) = 0$; and indeed, $r^2 - 2r + 1 = 0$. Observe that $r^2 - 2r + 1 = (r-1)^2$. Apply Fact 11.1 to see that $y = e^x$ and $y = xe^x$ BOTH are solutions of 11.1.1. Thus, the general solution of (11.1.1) is $y = c_1 e^x + c_2 x e^x$ for constants c_1 and c_2.

We verify that $y = c_1 e^x + c_2 x e^x$ really is a solution of (11.1.1). (This will provide a little evidence that Fact 11.1 is indeed correct.) If

$$y = c_1 e^x + c_2 x e^x,$$

then

$$\begin{aligned} y' &= c_1 e^x + c_2 x e^x + c_2 e^x, \\ y'' &= c_1 e^x + c_2 x e^x + 2c_2 e^x, \quad \text{and} \end{aligned}$$

$$\begin{aligned} y'' - 2y' + y &= (c_1 e^x + c_2 x e^x + 2c_2 e^x) - 2(c_1 e^x + c_2 x e^x + c_2 e^x) + (c_1 e^x + c_2 x e^x) \\ &= (c_1 - 2c_1 + c_1)e^x + (c_2 - 2c_2 + c_2)xe^x + (2c_2 - 2c - 2)e^x = 0. \checkmark \end{aligned}$$

11.B. **Suppose the characteristic equation has non-real roots.**

Fact 11.2. *Suppose you are trying to solve*

$$(11.0.2) \qquad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

and the corresponding characteristic polynomial

$$(11.0.3) \qquad a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$$

has a factor of the from $(r - (a + bi))^m$*, then the functions*

$$x^j e^{ax} \sin bx, \quad x^j e^{ax} \cos bx$$

all are solutions of (11.0.2)*, for* $0 \le j \le m - 1$.

[10]It is perfectly legal to jump right from the original Differential Equation to the Characteristic Equation without writing down y' and y''.

Example. Solve $y''+y=0$. We consider the characteristic polynomial r^2+1. Of course, $(r-i)(r+i)=r^2+1$. Apply Fact 11.2 to the factor $r-i$ of the characteristic polynomial. In other words, take $a=0$, $b=1$, and $m=1$. Conclude that $y=x^0e^{0x}\sin(1x)$ and $y=x^0e^{0x}\cos(1x)$ both are solutions of $y''+y=0$. (In other words, $y=\sin x$ and $y=\cos x$ both are solutions of $y''+y=0$. This is true and obvious.) Notice that the other factor, $r+i$, does not give anything more. Indeed, if we apply Fact 11.2 with $a=0$, $b=-1$ (because $r+i$ is the same as $r-(-i)$), and $m=1$, then Fact 11.2 yields that $y=x^0e^{0x}\sin(-1x)$ and $y=x^0e^{0x}\cos(-1x)$ both are solutions of $y''+y=0$. In other words, $y=-\sin x$ and $y=\cos x$ are solutions of $y''+y=0$. We already knew about these solutions.

Example. Solve $y^{(4)}+2y''+y=0$. We consider the characteristic polynomial
$$r^4+2r^2+1.$$
Of course,
$$r^4+2r^2+1=(r^2+1)^2=(r-i)^2(r+i)^2.$$
Apply Fact 11.2 to the factor $(r-i)^2$ of the characteristic polynomial. In other words, take $a=0$, $b=1$, and $m=2$. Conclude that
$$y=x^0e^{0x}\sin(1x),\quad y=x^0e^{0x}\cos(1x),\quad y=x^1e^{0x}\sin(1x),\quad \text{and}$$
$$y=x^1e^{0x}\cos(1x)$$
all are solutions of $y^{(4)}+2y''+y=0$. In other words,
$$y=\sin x,\quad y=\cos x,\quad y=x\sin x,\quad \text{and}\quad y=x\cos x$$
all are solutions of $y^{(4)}+2y''+y=0$. Notice that the other factor, $(r+i)^2$, does not give anything more. Indeed, if we apply Fact 11.2 with $a=0$, $b=-1$ (because $r+i$ is the same as $r-(-i)$), and $m=2$, then Fact 11.2 yields that
$$y=x^0e^{0x}\sin(-1x),\quad y=x^0e^{0x}\cos(-1x),\quad y=x^1e^{0x}\sin(-1x),\quad \text{and}$$
$$y=x^1e^{0x}\cos(-1x)$$
all are solutions of $y^{(4)}+2y''+y=0$. In other words,
$$y=-\sin x,\quad y=\cos x,\quad y=-x\sin x,\quad \text{and}\quad y=x\cos x$$
are solutions of $y^{(4)}+2y''+y=0$. We already knew about these solutions.

We check that $y=x\cos x$ really is a solution of $y^{(4)}+2y''+y=0$. Observe that
$$\begin{aligned}
y' &= -x\sin x+\cos x,\\
y'' &= -x\cos x-2\sin x,\\
y''' &= x\sin x-3\cos x,\\
y^{(4)} &= x\cos x+4\sin x, \text{and}\\
y^{(4)}+2y''+y &= \Big(x\cos x+4\sin x\Big)+2\Big(-x\cos x-2\sin x\Big)+\Big(x\cos x\Big)\\
&= (1-2+1)x\cos x+(4-4)\sin x=0.\checkmark
\end{aligned}$$

Example. Solve $y'' - 2y' + 5y = 0$.

The characteristic equation is $r^2 - 2r + 5 = 0$. Recall the quadratic formula. If $ax^2 + bx + c = 0$, then
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
If r is a root of the characteristic equation, then
$$r = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$
We apply Fact 11.2 with $a = 1$, $b = 2$ and $m = 1$ to see that
$$y = e^{ax}\cos(bx) \quad \text{and} y = e^{ax}\sin(bx)$$
are solutions of $y'' - 2y' + 5y = 0$. Thus, the general solution of $y'' - 2y' + 5y = 0$ is
$$y = c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

We check that $y = e^x \cos(2x)$ really is a solution of $y'' - 2y' + 5y = 0$. Observe that if $y = e^x \cos(2x)$, then
$$\begin{aligned}
y' &= -2e^x\sin(2x) + e^x\cos(2x)\\
y'' &= -2e^x\sin(2x) - 4e^x\cos(2x)\\
&\quad -2e^x\sin(2x) + e^x\cos(2x)\\
&= -4e^x\sin(2x) - 3e^x\cos(2x) \quad \text{and}\\
y'' - 2y' + 5y &= \Big(-4e^x\sin(2x) - 3e^x\cos(2x)\Big)\\
&\quad \Big(-2e^x\sin(2x) + e^x\cos(2x)\Big)(-2)\\
&\qquad + 5e^x\cos(2x)\\
&= (-4+4)e^x\sin(2x) + (-3-2+5)(e^x\cos(2x)) = 0.\checkmark
\end{aligned}$$

11.C. **Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$.**

We started the section by observing that the basic solutions of a linear homogeneous Differential Equation with constant coefficients, that is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{11.2.1}$$

where the a_j are real numbers, are exponential functions of the form $y = e^{rx}$. All of a sudden, Trig functions made an appearance. It is reasonable to ask how Trig functions are related to exponential functions. The answer is supplied by Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{11.2.2}$$

Once one has Euler's identity, then one knows that
$$e^{(a+bi)x} = e^{ax}e^{(bx)i} = e^{ax}(\cos(bx) + i\sin(bx))$$

and

$$e^{(a-bi)x} = e^{ax}e^{(-bx)i} = e^{ax}(\cos(bx) - i\sin(bx)).$$

So, in particular, if $a+bi$ is a root of the characteristic equation

$$a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0,$$

then $a-bi$ is also a root of the characteristic equation; hence,

$$e^{ax}(\cos(bx) + i\sin(bx)) \quad \text{and} \quad e^{ax}(\cos(bx) - i\sin(bx))$$

both are solutions of 11.2.1; hence

$$\frac{e^{ax}(\cos(bx) + i\sin(bx)) + e^{ax}(\cos(bx) - i\sin(bx))}{2} = e^{ax}\cos(bx)$$

and

$$\frac{e^{ax}(\cos(bx) + i\sin(bx)) - e^{ax}(\cos(bx) - i\sin(bx))}{2i} = e^{ax}\sin(bx)$$

are solutions of 11.2.1.

Euler's identity is a quick consequence of Taylor's series:[11]

$$\begin{aligned}
e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots, \\
\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots, \\
\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots,
\end{aligned}$$

for all complex numbers z. It follows that

$$\begin{aligned}
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\
&= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}$$

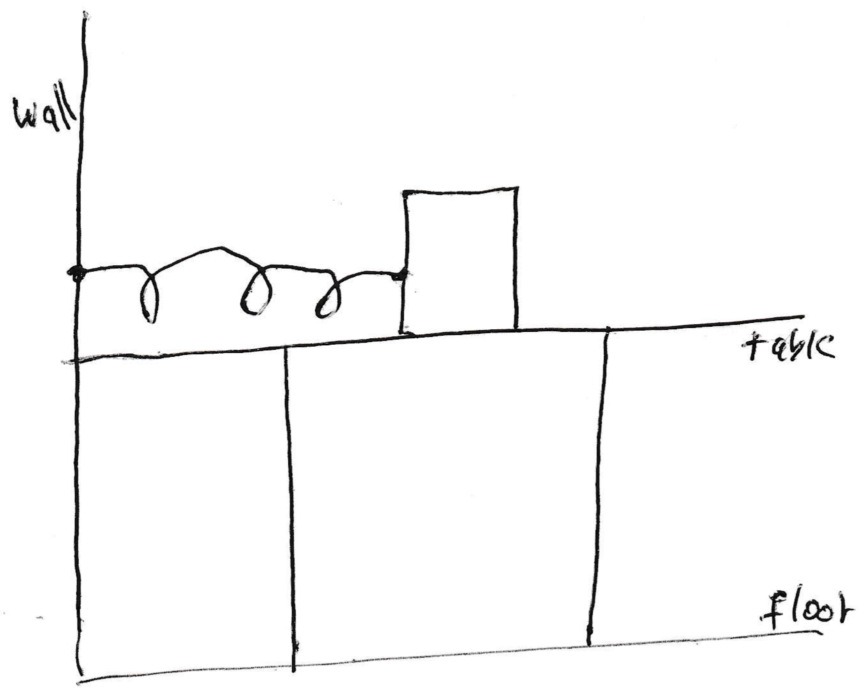
[11]In fact, I think that one studies Taylor's series in second semester calculus in order to make Euler's identity make sense to you in Differential Equations.

12. Section 3.4 Spring Problems.

Consider a spring with one end attached to a wall and the other end attached to a mass m which is sitting on a level surface. The spring constant is k. Assume that resistance is proportional to velocity. Let $x(t)$ be the displacement of the spring from its rest position. Give the Differential Equation for $x(t)$.

There is a picture of the setup on the next page.

Spring Problems



We are looking for a Differential Equation that determines $x(t)$. We use $F = ma$. There are two forces acting on the spring:

- the force of the spring and
- resistance

Hooke's Law says that the force exerted by the spring is proportional to and in the opposite direction. So the Spring force is $-kx$.

We have assumed that resistance is proportional to velocity. So the resistance force is $-cx'$.

Thus in this situation, Newton's Second Law of Motion, $F = ma$, becomes

$$mx'' = -cx' - kx$$

or $mx'' + cx' + kx = 0$, with all three coefficients m, c, and k positive. Of course, this is a second order homogeneous Differential Equation with constant coefficients. So there are three possibilities for the solution; either:

- $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$,or
- $x(t) = e^{rt}(c_1 + tc_2)$, or
- $x(t) = e^{at}(c_1 \sin bt + c_2 \cos bt)$

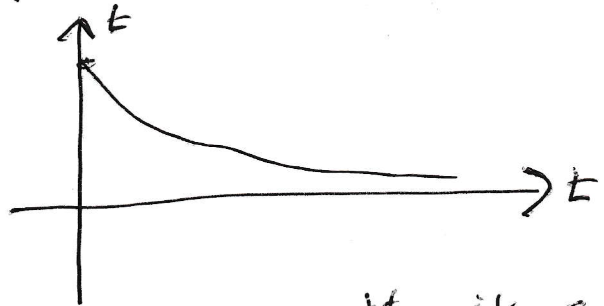
(In each case the r_1, r_2, r, and a is negative.) In the first two cases, the graph looks basically looks like $x = ce^{rt}$ with r negative. The third case can be transformed into $x = Ae^{at} cos(\omega t - \alpha)$. The graph bounces back and forth between $x = Ae^{at}$ and $x = -Ae^{at}$. This third case is the typical behavior of a spring from our observations.

There are pictures on the next page.

The Graph of $x(t)$.

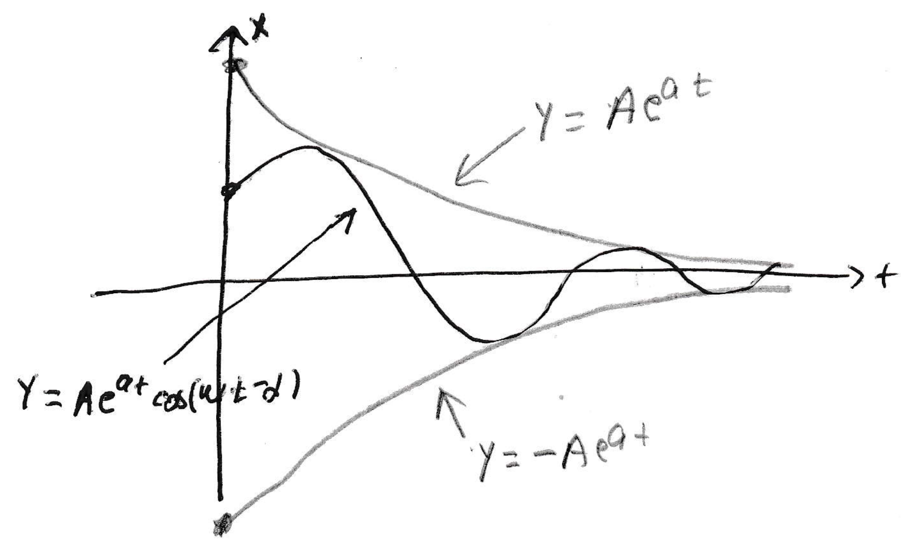
Let $x(t)$ be the displacement of a spring from its rest position. In the long run $x(t)$ looks much like $x = ce^{rt}$ or $x = Ae^{at}\cos(\omega t - \delta)$ with r and a negative

Here is $x = ce^{rt}$ with c positive



Here is $x = ce^{rt}$ with c negative

Here is $x = Ae^{at}\cos(\omega t - \delta)$ with A positive



12.A. **We transform $c_1\cos(bt)+c_2\sin(bt)$ into $A\cos(\omega t-\alpha)$.**

Recall that $\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$. You might remember this identity from your Trig class or maybe a calculus class; but now you have a powerful new way to understand (and remember) this identity. Use Euler's identity (and multiplication of complex numbers) to see that

$$\begin{aligned}&\cos(\theta-\phi)+i\sin(\theta-\phi)=e^{i(\theta-\phi)}=e^{i\theta}e^{i(-\phi)}\\ =&\Big(\cos(\theta)+i\sin(\theta)\Big)\Big(\cos(-\phi)+i\sin(-\phi)\Big)\\ =&\Big(\cos(\theta)+i\sin(\theta)\Big)\Big(\cos(\phi)-i\sin(\phi)\Big)\\ =&\Big(\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi)\Big)+i\Big(\sin(\theta)\cos(\phi)-\cos(\theta)\sin(\phi)\Big).\end{aligned}$$

Equate the real part of the expression at the beginning with the real part of the expression at the end and equate the imaginary part of the expression at the beginning with the imaginary part of the expression at the end to see that

$$\begin{aligned}\cos(\theta-\phi)&=\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi)\\ \sin(\theta-\phi)&=\sin(\theta)\cos(\phi)-\cos(\theta)\sin(\phi)\end{aligned}$$

We wanted the first formula, but we got the second one for free.

At any rate, we want to write

$$(\cos(bt))c_1+(\sin(bt))c_2$$

in the form

$$A\cos(bt-\alpha)=A\Big(\cos(bt)\cos(\alpha)+\sin(bt)\sin(\alpha)\Big)$$

If it made sense, we would pick α with $\cos(\alpha)=c_1$ and $\sin(\alpha)=c_2$. Of course, **this does not make sense** because the distance from to origin to the point

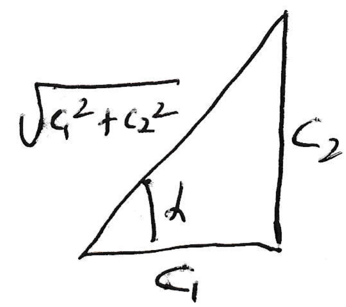
$$(\cos(\alpha),\sin(\alpha))$$

is 1 and the distance from the origin to (c_1,c_2) is what ever it is. But this thought gives us the right idea. We should think about the triangle with

$$\text{ADJ}=c_1,\quad \text{OP}=c_2,\quad \text{and}\quad \text{HYP}=\sqrt{c_1^2+c_2^2}.$$

To convert $(\cos bt)c_1 + (\sin bt)c_2$ into $A\cos(bt-\alpha)$

Consider the triangle



observe that

$(\cos bt)c_1 + (\sin bt)\,c_2$

$$= \sqrt{c_1^2+c_2^2}\left((\cos bt)\frac{c_1}{\sqrt{c_1^2+c_2^2}} + (\sin bt)\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right)$$

$$= \sqrt{c_1^2+c_2^2}\left((\cos bt)\cos\alpha + (\sin bt)\sin\alpha\right)$$

use $\cos\alpha = \frac{ADJ}{HYP} = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\sin\alpha = \frac{OP}{HYP} = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$

$$= \sqrt{c_1^2+c_2^2}\,\cos(bt-\alpha)$$

for $\alpha = \arccos \frac{c_1}{\sqrt{c_1^2+c_2^2}}$

Let $\alpha = \arccos\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right)$ and $A = \sqrt{c_1^2+c_2^2}$. Observe that

$$\begin{aligned}
&(\cos(bt))c_1 + (\sin(bt))c_2 \\
&= \sqrt{c_1^2+c_2^2}\left(\cos(bt))\frac{c_1}{\sqrt{c_1^2+c_2^2}} + (\sin(bt))\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right) \\
&= \sqrt{c_1^2+c_2^2}\Big(\cos(bt)\cos(\alpha) + \sin(bt)\sin(\alpha)\Big) \\
&= A\cos(bt-\alpha).
\end{aligned}$$

Example. (This is number 20 from section 3.4.) Solve the Initial Value Problem

$$2x'' + 16x' + 40x = 0, \quad x(0) = 5, \quad x'(0) = 4.$$

Put the answer in the form

$$x(t) = Ae^{at}\cos(bt-\alpha)$$

if this makes sense.

We get to work. It does no harm to divide both sides of the Differential Equation by 2. We study

$$x'' + 8x' + 20x = 0.$$

The characteristic equation is

$$r^2 + 8r + 20 = 0.$$

Use the quadratic formula to see that the roots of the characteristic equation are

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a},$$

where $a = 1$, $b = 8$, and $c = 20$. Thus, the roots of the characteristic equation are

$$r = \frac{-8 \pm \sqrt{64-80}}{2} = \frac{-8 \pm 4i}{2} = -4 \pm 2i.$$

So the general solution of the DE is

$$x = e^{-4t}(c_1\cos(2t) + c_2\sin(2t)).$$

We compute

$$x' = e^{-4t}(-2c_1\sin(2t) + 2c_2\cos(2t)) - 4e^{-4t}(c_1\cos(2t) + c_2\sin(2t))$$

Plug in $t = 0$ to see that

$$5 = x(0) = c_1 \quad \text{and} \quad 4 = x'(0) = 2c_2 - 4c_1.$$

It follows that $c_1 = 5$ and $c_2 = 12$. Thus,

$$x(t) = e^{-4t}\Big(\big(\cos(2t)\big)5 + \big(\sin(2t)\big)12\Big).$$

Recall that $\sqrt{5^2+12^2} = 13$. So

$$x(t) = 13e^{-4t}\Big(\big(\cos(2t)\big)\tfrac{5}{13} + \big(\sin(2t)\big)\tfrac{12}{13}\Big).$$

Let $\alpha = \arccos\frac{5}{13}$. We have found that

$$\boxed{x(t) = 13e^{-4t}(\cos(2t - \arccos\tfrac{5}{13})}.$$

13. Section 3.5: The method of the undetermined coefficient.

In this section we describe a method for finding a particular solution for

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = q(x),$$

where the a_i's are constants.

The method is called "The method of the undetermined coefficient". One guesses the form of the solution and adjusts the constants.

Example. Find one particular solution for Differential Equations (a), (b), (c), and (d). Find the general solution of (e).

(a) $y'' + 2y' - 3y = 6x$
(b) $y'' + 2y' - 3y = 2\sin 2x$
(c) $y'' + 2y' - 3y = e^{2x}$
(d) $y'' + 2y' - 3y = e^{x}$
(e) $y'' - 4y' + 4y = xe^{2x}$

(a) We try $y = Ax + B$. Observe that if $y = Ax + B$, then $y' = A$, $y'' = 0$, and

$$y'' + 2y' - 3y = 6x$$

if and only if

$$0 + 2A - 3(Ax + B) = 6x.$$

Rewrite the most recent equation as

$$(-3A)x + (2A - 3B) = 6x.$$

Our y satisfies the DE provided

$$-3A = 6 \quad \text{and} \quad 2A - 3B = 0.$$

We take $A = -2$ and $B = \frac{-4}{3}$. One particular solution of (a) is $\boxed{y_{\text{partic}} = -2x - \frac{4}{3}.}$

(b) We try $y = A\sin 2x + B\cos 2x$. If $y = A\sin 2x + B\cos 2x$, then

$$y' = 2A\cos 2x - 2B\sin 2x$$

and

$$y'' = -4A\sin 2x - 4B\cos 2x.$$

We see that $y'' + 2y' - 3y = 2\sin 2x$ if and only if

$$(-4A\sin 2x - 4B\cos 2x) + 2(2A\cos 2x - 2B\sin 2x) - 3(A\sin 2x + B\cos 2x) = 2\sin 2x.$$

Rewrite the most recent equation as

$$(-4A - 4B - 3A)\sin 2x + (-4B + 4A - 3B) = 2\sin 2x.$$

Our y satisfies the DE provided

$$-7A - 4B = 2 \quad \text{and} \quad 4A - 7B = 0.$$

I will write this system of equations as a matrix equation and multiply by the inverse matrix.

$$\begin{bmatrix} -7 & -4 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{13.0.1}$$

The inverse of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

I multiply both sides (13.0.1) on the left by

$$\frac{1}{49+16}\begin{bmatrix} -7 & 4 \\ -4 & -7 \end{bmatrix}$$

to obtain

$$\frac{1}{65}\begin{bmatrix} -7 & 4 \\ -4 & -7 \end{bmatrix}\begin{bmatrix} -7 & -4 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{65}\begin{bmatrix} -7 & 4 \\ -4 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This equation gives

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{65}\begin{bmatrix} -14 \\ -8 \end{bmatrix}.$$

We conclude that one particular solution of (b) is $\boxed{y_{\text{partic}} = \frac{-14}{65}\sin 2x - \frac{8}{65}\cos 2x.}$

(c) We look for a particular solution of $y'' + 2y' - 3y = e^{2x}$. We try $y = Ae^{2x}$. We compute $y' = 2Ae^{2x}$ and $y'' = 4Ae^{2x}$. The function $y = Ae^{2x}$ is a solution of

$$y'' + 2y' - 3y = e^{2x}$$

provided

$$\begin{gathered} 4Ae^{2x} + 2(2Ae^{2x}) - 3Ae^{2x} = e^{2x} \\ 5Ae^{2x} = e^{2x} \end{gathered}$$

We have calculated that if $A = \frac{1}{5}$, then $y = Ae^{2x}$ is a solution of

$$y'' + 2y' - 3y = e^{2x}.$$

We conclude that

$$\boxed{y = \tfrac{1}{5}e^{2x} \text{ is a solution of } y'' + 2y' - 3y = e^{2x}.}$$

(d) We look for a particular solution of $y'' + 2y' - 3y = e^{x}$. We try $y = Ae^{x}$. We compute $y' = Ae^{x}$ and $y'' = Ae^{x}$. The function $y = Ae^{x}$ is a solution of

$$y'' + 2y' - 3y = e^{x}$$

provided

$$Ae^{x} + 2Ae^{x} - 3Ae^{x} = e^{x}.$$

The most recent equation is $0 = e^{x}$. Of course this does not ever happen. We conclude that $y = Ae^{x}$ is never a solution of $y'' + 2y' - 3y = e^{x}$. Of course,

$y = Ae^x$ is never a solution of $y'' + 2y' - 3y = e^x$ because $y = Ae^x$ is always a solution of $y'' + 2y' - 3y = 0$. Hmm. So what should we do? We remember that when a characteristic polynomial has a factor $x - r$ of multiplicity two, we found "the missing" solution of the corresponding Homogeneous Linear DE by considering $y = xe^{rx}$. Let us try the same trick here. Let us see if there exists a constant A with $y = Axe^x$ a solution of $y'' + 2y' - 3y = e^x$. We compute $y' = Axe^x + Ae^x$ and $y'' = Axe^x + 2Ae^x$. The function $y = Axe^x$ is a solution of

$$y'' + 2y' - 3y = e^x$$

provided

$$(Axe^x + 2Ae^x) + 2(Axe^x + Ae^x) - 3(Axe^x) = e^x.$$

This is the same as

$$(A + 2A - 3A)xe^x + (2A + 2A)e^x = e^x$$

or

$$4Ae^x = e^x.$$

We conclude that

$$\boxed{y = \tfrac{1}{4}xe^x \text{ is a solution of } y'' + 2y' - 3y = e^x.}$$

(e) We look for general solution of $y'' - 4y' + 4y = xe^{2x}$.

The characteristic polynomial is $r^2 - 4r + 4 = (r - 2)^2$. So, the general solution of the homogeneous problem $y'' - 4y' + 4y = 0$ is $y = c_1e^{2x} + c_2xe^{2x}$.

If we were trying to find a particular solution for $y'' - 4y' + 4y = e^{2x}$, it would not make sense to look for a solution of the form $y = Ae^{2x}$, because all such functions are solutions of the homogeneous problem $y'' - 4y' + 4y = 0$. Similarly, it would not make sense to look for a particular solution for $y'' - 4y' + 4y = e^{2x}$ of the form $y = Axe^{2x}$, because all such functions are solutions of the homogeneous problem $y'' - 4y' + 4y = 0$. It would make sense to look for a particular solution for $y'' - 4y' + 4y = e^{2x}$ of the form $y = Ax^2e^{2x}$.

In fact, we want a particular solution of $y'' - 4y' + 4y = xe^{2x}$. It does not make sense to look for a solution of the form $y = Ae^{2x}$, or $y = Axe^{2x}$, or $y = Ax^2e^{2x}$ because these functions all satisfy some other differential equation. It does make sense to look for a solution of the form $y = Ax^3e^{2x}$. So, lets do it!!

Take $y = Ax^3e^{2x}$. Calculate

$$\begin{aligned} y' &= (2Ax^3 + 3Ax^2 \qquad\quad)e^{2x} \\ y'' &= (4Ax^3 + 12Ax^2 + 6Ax)e^{2x} \end{aligned}$$

Thus $y = Ax^3e^{2x}$ is a solution of $y'' - 4y' + 4y = xe^{2x}$ provided

$$(4Ax^3 + 12Ax^2 + 6Ax)e^{2x} - 4(2Ax^3 + 3Ax^2)e^{2x} + 4Ax^3e^{2x} = xe^{2x}.$$

The most recent equation is

$$A\Big((4-8+4)x^3+(12-12)x^2+6x\Big)e^{2x}=xe^{2x}$$

or

$$6Axe^{2x}=xe^{2x}.$$

We have calculated that $y=\frac{1}{6}x^3e^{2x}$ is a particular solution of $y''-4y'+4y=xe^{2x}$ and

$$\boxed{y=c_1e^{2x}+c_2xe^{2x}+\tfrac{1}{6}x^3e^{2x} \text{ is the general solution of } y''-4y'+4y=xe^{2x}.}$$

14. Section 7.1: Laplace Transforms

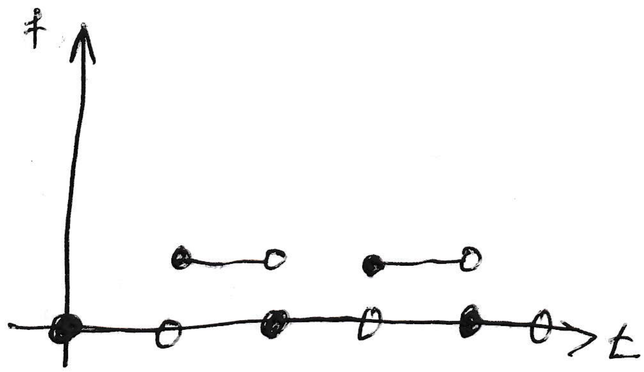
Why? One can use Laplace Transforms to solve Initial Value Problems. Laplace Transforms are especially good for solving Initial Value Problems of the form

$$\begin{cases} mx'' + cx' + kx = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

where $f(t)$ is piece-wise continuous, or continuous and piece-wise differentiable.

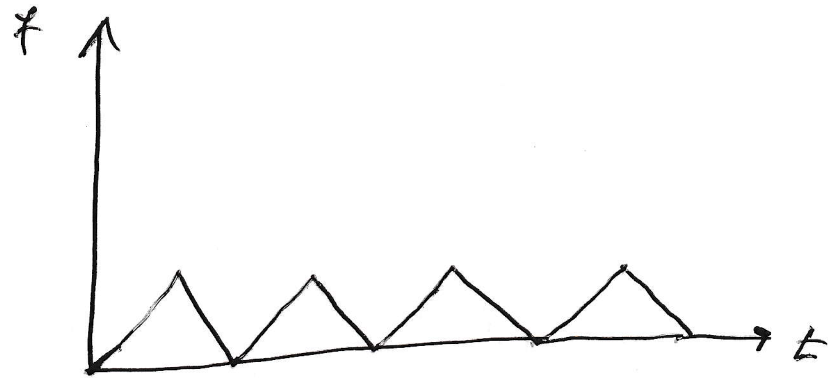
There is a picture of a piece-wise continuous function on the next page. There is also a picture of a continuous, but piece-wise differentiable, function on the next page.

The function



is piece-wise continuous

The function



is continuous and piecewise differentiable.

What? If $f(t)$ is a function, then $\mathscr{L}(f)$ is a new function F in a new variable s: $\mathscr{L}(f) = F(s)$ with

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt. \tag{14.0.1}$$

Here is the plan for section 7.1.

(1) We fill in the chart

f	$\mathscr{L}(f) = F(s)$	restriction on s
1		
e^{at}		
t		
$\sin(at)$		
$\cos(at)$		

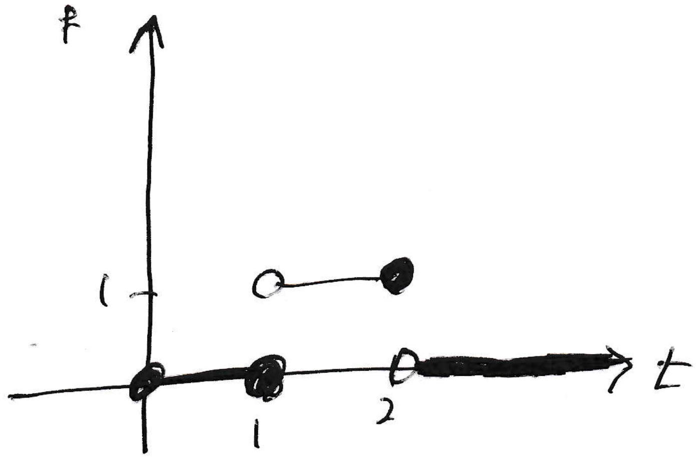
(2) We do problem 8 from 7.1. That is, we calculate $\mathscr{L}(f)$ for

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le 1 \\ 1 & \text{for } 1 < t \le 2 \\ 0 & \text{for } 2 < t. \end{cases}$$

There is a picture of $f(t)$ on the next page.

f(t) from Problem 8 from 7.1

$$f(t) = \begin{cases} 0 & \text{for } 0 \le t \le 1 \\ 1 & \text{for } 1 < t \le 2 \\ 0 & \text{for } 2 < t. \end{cases}$$



(3) We do problem 18 from 7.1. That is, we calculate $\mathscr{L}(f)$ for

$$f(t) = \sin(3t)\cos(3t).$$

(4) (a) Find $f(t)$ with $\mathscr{L}(f) = F(s)$ with $F(s) = \frac{1}{s+5}$.
(b) Find $f(t)$ with $\mathscr{L}(f) = F(s)$ with $F(s) = \frac{3s+1}{s^2+4}$.

Calculation 14.1. We calculate

$$\mathscr{L}(1) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st}dt = \lim_{w\to\infty} \frac{-1}{s} e^{-st}\Big|_0^w = \lim_{w\to\infty} \left(\frac{-1}{s}e^{-sw} + \frac{1}{s}\right).$$

Observe that if $0 < s$, then $\lim_{w\to\infty} e^{-sw} = 0$. Thus,

$$\boxed{\mathscr{L}(1) = \frac{1}{s}, \text{ provided } 0 < s.}$$

Calculation 14.2. We calculate

$$\mathscr{L}(e^{at}) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \lim_{w\to\infty} \frac{-1}{s-a} e^{-(s-a)t}\Big|_0^w$$
$$= \lim_{w\to\infty} \left(\frac{-1}{s-a}e^{-(s-a)w} + \frac{1}{s-a}\right).$$

Observe that if $a < s$, then $\lim_{w\to\infty} e^{-(s-a)w} = 0$. Thus,

$$\boxed{\mathscr{L}(e^{at}) = \frac{1}{s-a}, \text{ provided } a < s.}$$

Remark. Of course, Calculation 14.1 is the special case of Calculation 14.2 where a where is taken to be zero.

Calculation 14.3. We calculate

$$\mathscr{L}(t) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st}t dt.$$

We use integration by parts which says that

$$\int u dv = uv - \int v du.$$

Take $u = t$ and $dv = e^{-st}dt$. Calculate $du = dt$ and $v = \frac{1}{-s}e^{-st}$. It follows that

$$\int e^{-st}t dt = \frac{t}{-s}e^{-st} - \int \frac{1}{-s}e^{-st}dt = \frac{t}{-s}e^{-st} - \frac{1}{s^2}e^{-st}.$$

It follows that

$$\mathscr{L}(t) = \lim_{w\to\infty} \left[\frac{t}{-s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^w$$
$$= \lim_{w\to\infty} \left[\frac{w}{-s}e^{-sw} - \frac{1}{s^2}e^{-sw} - 0 + \frac{1}{s^2}\right]$$

Notice that if $0 < s$, then $\lim_{w\to\infty} we^{-sw} = 0$ and $\lim_{w\to\infty} e^{-sw} = 0$. We conclude that

$$\boxed{\mathscr{L}(t) = \frac{1}{s^2}, \text{ provided } 0 < s}$$

Remark. Calculation 14.3 was amazingly complicated. We will introduce the Gamma function to calculate the Laplace Transform of other powers of t by way of trickery. Speaking of trickery, the integral of $e^{-st}\sin(at)$ (with respect to t) is doable but very unpleasant. (Maybe you remember this integral from second semester calculus). So we use Euler's identity to write $\sin(at)$ in terms of (complex) exponential functions. This is a brilliant idea because it is easy to compute the Laplace Transform of an exponential function. Indeed, if a is a complex number $\mathscr{L}(e^{at}) = \frac{1}{s-a}$ **provided s is greater than the real part of a.**

Calculation 14.4. We compute $\mathscr{L}(\sin(at))$. Recall that

$$\begin{aligned} e^{ati} &= \cos(at) + i\sin(at) \\ e^{-ati} &= \cos(at) - i\sin(at) \end{aligned}$$

Thus,

$$\frac{e^{ati} - e^{-ati}}{2i} = \sin(at).$$

It follows that

$$\mathscr{L}(\sin(at)) = \mathscr{L}\left(\frac{e^{ati} - e^{-ati}}{2i}\right)$$

It is easy to see that $\mathscr{L}(af + bg) = a\mathscr{L}(f) + b\mathscr{L}(g)$ where a and b are constants and f and g are functions.

$$\begin{aligned} &= \frac{1}{2i}\Big(\mathscr{L}(e^{ait}) - \mathscr{L}(e^{-ait})\Big) \\ &= \frac{1}{2i}\Big(\frac{1}{s-ai} - \frac{1}{s+ai}\Big) \\ &= \frac{1}{2i}\Big(\frac{s+ai-(s-ai)}{(s-ai)(s+ai)}\Big) \\ &= \frac{1}{2i}\Big(\frac{2ai}{s^2+a^2}\Big) \\ &= \frac{a}{s^2+a^2}, \end{aligned}$$

provided s is larger than the real part of ai. In other words,

$$\boxed{\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2}, \text{ provided } 0 < s.}$$

Calculation 14.5. We compute

$$\begin{aligned}\mathscr{L}(\cos(at)) &= \mathscr{L}(e^{ati} - i\sin(at))\\ &= \mathscr{L}(e^{ati}) - i\mathscr{L}(\sin(at))\\ &= \frac{1}{s-ai} - \frac{ia}{s^2+a^2}\\ &= \frac{s+ai}{s^2+a^2} - \frac{ia}{s^2+a^2}\\ &= \frac{s}{s^2+a^2}.\end{aligned}$$

$$\boxed{\mathscr{L}(\cos(at)) = \frac{s}{s^2+a^2},\ \text{provided } 0 < s.}$$

We can now fill in the chart from (1) on page 65

(14.5.1)

f	$\mathscr{L}(f) = F(s)$	restriction on s
1	$\frac{1}{s}$	$0 < s$
e^{at}	$\frac{1}{s-a}$	$a < s$
t	$\frac{1}{s^2}$	$0 \leq s$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$0 \leq s$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$0 \leq s$

• Now we look at item (2) from page 65.

We compute $\mathscr{L}(f(t))$ for

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 1\\ 1 & \text{for } 1 < t \leq 2\\ 0 & \text{for } 2 < t.\end{cases}$$

To do this, we just follow our noses. Recall from (14.0.1) on page 65 that

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

It follows that for the $f(t)$ of this problem,

$$\mathscr{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = \begin{cases} \int_0^1 e^{-st} f(t)dt\\ +\int_1^2 e^{-st} f(t)dt\\ +\int_2^\infty e^{-st} f(t)dt\end{cases}$$

$$= \left\{\begin{array}{l} \int_0^1 e^{-st} 0dt\\ +\int_1^2 e^{-st} 1dt\\ +\int_2^\infty e^{-st} 0dt\end{array}\right\} = \int_1^2 e^{-st}dt = \frac{1}{-s}e^{-st}\Big|_1^2 = \boxed{\frac{1}{-s}(e^{-2s} - e^{-s}).}$$

• Now we look at item (3) from page 67.

We compute $\mathscr{L}(\sin(3t)\cos(3t))$. Recall that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. (Maybe you remember this from Trig or Calculus. If not, get it from Euler's Identity as we did in section 12.A on page 55.)

Use the trig identity (with θ replaced by $3t$) and the Table 14.5.1 from page 69 to see that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta);$$

hence

$$\sin(6t) = 2\sin(3t)\cos(3t).$$

It follows that

$$\mathscr{L}(\sin(3t)\cos(3t)) = \mathscr{L}\left(\frac{\sin(6t)}{2}\right) = \frac{1}{2}\mathscr{L}\left(\sin(6t)\right) = \frac{1}{2}\frac{6}{s^2+36} = \boxed{\frac{3}{s^2+36}.}$$

• Now we look at item (4) from page 67.

(a) Find $f(t)$ with $\mathscr{L}(f) = F(s)$ with $F(s) = \frac{1}{s+5}$.

(b) Find $f(t)$ with $\mathscr{L}(f) = F(s)$ with $F(s) = \frac{3s+1}{s^2+4}$.

Look at Table 14.5.1 on page 69

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}.$$

Take $a = -5$ to see that

$$\mathscr{L}(e^{-5t}) = \frac{1}{s+5}.$$

So,

$$\boxed{\mathscr{L}^{-1}\left(\frac{1}{s+5}\right) = e^{-5t}}.$$

Look at Table 14.5.1 on page 69

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2} \quad \text{and} \quad \mathscr{L}(\cos(at)) = \frac{s}{s^2+a^2}.$$

It follows that

$$\mathscr{L}^{-1}(\frac{3s+1}{s^2+4}) = 3\mathscr{L}^{-1}(\frac{s}{s^2+4}) + \frac{1}{2}\mathscr{L}^{-1}(\frac{2}{s^2+4}) = \boxed{3\cos(2t) + \frac{1}{2}\sin(2t).}$$

15. Sections 7.2 and 7.3: We use Laplace Transforms to solve Initial Value Problems.

We have calculated

f	$\mathscr{L}(f)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$

I distributed the Laplace Transform facts that I will give you when you take the final. There are a few copies up front if you want another one. They are also available on the class website.

Today's agenda:

- There are three facts in these sections.
- Use Laplace transforms to solve $x''+4x=\sin 2t,\ x(0)=x'(0)=0$. Use HW 31 in 7.2.
- Find $\mathscr{L}(t\sin kt)$ (Ex 5 in 7.2 in the book.)
- Find $\mathscr{L}(t\cos kt)$ (HW 28 in 7.2)
- Find $\mathscr{L}^{-1}\left(\frac{2s-3}{9s^2-12s+20}\right)$ (7.3/10)
- Use Laplace transforms to solve $x''+6x'+18x=\cos 2t,\ x(0)=1,\ x'(0)=-1$.
- $\Gamma(x)$ from 7.1
- Why the three facts are true.

Fact 15.1. $\mathscr{L}(f')=s\mathscr{L}(f)-f(0)$

Fact 15.2. $\mathscr{L}^{-1}\left(\dfrac{F(s)}{s}\right)=\displaystyle\int_0^t [\mathscr{L}^{-1}(F(s))]\Big|_\tau d\tau.$

Fact 15.3. *If* $\mathscr{L}(f(t))=F(s)$, *then* $\mathscr{L}(e^{at}f(t))=F(s-a)$

Example. Use Laplace transform and the result of Homework problem 31 in section 7.2 to solve the Initial Value Problem

$$x''+4x=\sin(2t),\quad x(0)=x'(0)=0.$$

Let $X=\mathscr{L}(x)$. Apply Fact 15.1 to see that

$$\mathscr{L}(x')=sX\quad\text{and}\quad\mathscr{L}(x'')=s^2X.$$

We calculated $\mathscr{L}(\sin(2t))=\frac{2}{s^2+4}$ in Table 14.5.1. Apply $\mathscr{L}$ to $x''+4x=\sin(2t)$ to obtain

$$s^2X+4X=\frac{2}{s^2+4}$$

$$(s^2+4)X = \frac{2}{s^2+4}$$

$$X = \frac{2}{(s^2+4)^2}.$$

In Homework problem 31, you will calculate that

$$\mathscr{L}^{-1}\left(\frac{1}{(s^2+k^2)^2}\right) = \frac{1}{2k^3}(\sin(kt) - kt\cos(kt)). \tag{15.3.1}$$

In the present problem,

$$x = \mathscr{L}^{-1}(X) = 2\mathscr{L}^{-1}\left(\frac{1}{(s^2+4)^2}\right).$$

Apply Homework problem 31 with $k=2$ to see that

$$x = 2\frac{1}{2(8)}(\sin(2t) - 2t\cos(2t)).$$

The solution of the Initial Value Problem is

$$\boxed{x(t) = \tfrac{1}{8}\sin(2t) - \tfrac{1}{4}t\cos(2t).}$$

Check. We compute

$$x'(t) = \tfrac{1}{4}\cos(2t) - \tfrac{1}{4}\cos(2t) + \tfrac{1}{2}t\sin(2t) = \tfrac{1}{2}t\sin(2t)$$

$$x''(t) = \tfrac{1}{2}\sin(2t) + t\cos(2t)$$

We see that

$$x'' + 4x = (\tfrac{1}{2}\sin(2t) + t\cos(2t)) + 4(\tfrac{1}{8}\sin(2t) - \tfrac{1}{4}t\cos(2t))$$

$$= (\frac{1}{2} + \frac{1}{2})\sin(2t) + \cos(2t) - \cos(2t) = \sin(2t)\checkmark.$$

$$x(0) = \tfrac{1}{8}\sin(0) - \tfrac{1}{4}0(\cos(0)) = 0\checkmark$$

$$x'(0) = \tfrac{1}{2}0(\sin(0)) = 0\checkmark$$

Remark. Problem 31 is easy once one knows
(a) $\mathscr{L}(t\cos(kt)) = \frac{s^2-k^2}{(s^2+k^2)^2}$ and
(b) $\mathscr{L}(\sin(kt)) = \frac{k}{s^2+k^2}$.
Of course, we know (b). Indeed, this result is in Table 14.5.1. One uses trickery to get (a). This is problem 28. I did not assign 28; but I will do it in a minute. The trick for problem 28 is the same as the trick in Example 5 in the text book. So here is the plan. I will do Example 5 from the text book. Then I will do homework problem 28. I leave you with problem 31. (But you don't have to do any heavy lifting.)

Example 15.4. We compute $\mathscr{L}(t\sin(kt))$. I do not want to compute

$$\mathscr{L}(t\sin(kt)) = \int_0^\infty e^{-st} t\sin(kt)dt$$

directly. It looks much too hard. On the other hand, if $f(t) = t\sin(kt)$, then

$$f''(t) = -f(t) + \text{some more stuff};$$

so

$$\mathscr{L}(f''(t)) = -\mathscr{L}(f(t)) + \mathscr{L}(\text{some more stuff})$$

and Fact 15.1 tells us how $\mathscr{L}(f''(t))$ is related to $\mathscr{L}(f(t))$. If we are lucky, when can solve the equation

$$\mathscr{L}(f(t)) + \mathscr{L}(whatever) = -\mathscr{L}(f(t)) + \mathscr{L}(\text{some more stuff})$$

in order to find $\mathscr{L}(f(t))$.

We get to work. If $f(t) = t\sin(kt)$, then

$$f'(t) = kt\cos(kt) + \sin(kt)$$

and

$$f''(t) = -k^2 t\sin(kt) + 2k\cos(kt).$$

Read the most recent line as

$$f''(t) = -k^2 f(t) + 2k\cos(kt). \tag{15.4.1}$$

Recall from Fact 15.1 that

$$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$$

For us, $f(0) = 0$ and $\mathscr{L}(f') = s\mathscr{L}(f)$. Similarly,

$$\mathscr{L}(f'') = s\mathscr{L}(f') - f'(0)$$

with $f'(0) = 0$. Thus,

$$\mathscr{L}(f'') = s\mathscr{L}(f') - f'(0) = s^2\mathscr{L}(f)$$

Apply $\mathscr{L}$ to both sides of (15.4.1) to obtain

$$s^2\mathscr{L}(f) = -k^2\mathscr{L}(f) + 2k\mathscr{L}(\cos(kt)). \tag{15.4.2}$$

We know $\mathscr{L}(\cos(kt)) = \frac{s}{s^2+k^2}$. Solve equation (15.4.2) for $\mathscr{L}(f)$. Conclude

$$\boxed{\mathscr{L}(t\sin(kt)) = \frac{2ks}{(s^2+k^2)^2}.}$$

Example. We use the trick of Example 15.4 to find $\mathscr{L}(t\cos(kt))$. Let

$$f(t) = t\cos(kt).$$

Compute

$$f'(t) = -kt\sin(kt) + \cos(kt)$$

and

$$f''(t) = -k^2 t\cos(kt) - 2k\sin(kt). \tag{15.4.3}$$

Apply Fact 15.1 to see that

$$\mathscr{L}(f'(t)) = s\mathscr{L}(f(t)) - f(0) = s\mathscr{L}(f(t))$$

$$\mathscr{L}(f''(t)) = s\mathscr{L}(f'(t)) - f'(0) = s^2\mathscr{L}(f(t)) - 1$$

Rewrite equation (15.4.3) as

$$f''(t) = -k^2 f(t) - 2k\sin(kt)$$

and apply $\mathscr{L}$ to both sides

$$s^2\mathscr{L}(f(t)) - 1 = -k^2\mathscr{L}(f(t)) - 2k\mathscr{L}(\sin(kt)).$$

Use Table 14.5.1 and solve for $\mathscr{L}(f(t))$ to obtain

$$(s^2+k^2)\mathscr{L}(f(t)) = 1 - 2k\frac{k}{(s^2+k^2)}$$

$$\mathscr{L}(f(t)) = \frac{1}{s^2+k^2} - \frac{2k^2}{(s^2+k^2)^2}$$

$$\mathscr{L}(f(t)) = \frac{s^2-k^2}{(s^2+k^2)^2}.$$

Conclude

$$\boxed{\mathscr{L}(t\cos(kt)) = \frac{s^2-k^2}{(s^2+k^2)^2}.}$$

Example. Find $\mathscr{L}^{-1}\left(\frac{2s-3}{9s^2-12s+20}\right)$.

One might hope to factor the denominator and then apply the technique of partial fractions as was done in Example 15.5 on page 82. Unfortunately, the denominator does not factor (over the real numbers) because $b^2-4ac = (12)^2 - 4(9)(20) < 0$. Instead, we complete the square because we can use Fact 15.3 and Table 14.5.1 to compute

$$\mathscr{L}^{-1}\left(\frac{1}{(s-a)^2+k^2}\right) \quad \text{and} \quad \mathscr{L}^{-1}\left(\frac{s-a}{(s-a)^2+k^2}\right).$$

We get to work.

$$
\begin{aligned}
&\mathscr{L}^{-1}\left(\frac{2s-3}{9s^2-12s+20}\right)\\
&=\mathscr{L}^{-1}\left(\frac{2s-3}{9(s^2-\frac{4}{3}s)+20}\right)\\
&=\mathscr{L}^{-1}\left(\frac{2s-3}{9(s^2-\frac{4}{3}s+\frac{4}{9})+20-9(\frac{4}{9})}\right)\\
&=\mathscr{L}^{-1}\left(\frac{2s-3}{9(s-\frac{2}{3})^2+16}\right)\\
&=\mathscr{L}^{-1}\left(\frac{2(s-\frac{2}{3}+\frac{2}{3})-3}{9(s-\frac{2}{3})^2+16}\right)\\
&=\frac{2}{9}\mathscr{L}^{-1}\left(\frac{(s-\frac{2}{3})}{(s-\frac{2}{3})^2+\frac{16}{9}}\right)+\mathscr{L}^{-1}\left(\frac{-\frac{5}{3}}{9(s-\frac{2}{3})^2+16}\right)\\
&=\frac{2}{9}\mathscr{L}^{-1}\left(\frac{(s-\frac{2}{3})}{(s-\frac{2}{3})^2+\frac{16}{9}}\right)+\frac{1}{9}\left(-\frac{5}{3}\right)\frac{3}{4}\mathscr{L}^{-1}\left(\frac{\frac{4}{3}}{(s-\frac{2}{3})^2+\frac{16}{9}}\right)\\
&=\tfrac{2}{9}e^{\frac{2}{3}t}\cos(\tfrac{4}{3}t)-\tfrac{5}{36}e^{\frac{2}{3}t}\sin(\tfrac{4}{3}t).
\end{aligned}
$$

We conclude that

$$\boxed{\mathscr{L}^{-1}\left(\tfrac{2s-3}{9s^2-12s+20}\right)=\tfrac{2}{9}e^{\frac{2}{3}t}\cos(\tfrac{4}{3}t)-\tfrac{5}{36}e^{\frac{2}{3}t}\sin(\tfrac{4}{3}t).}$$

If you care, this is problem 10 in section 7.3.

Example. Use Laplace transforms to solve the Initial Value Problem

$$x''+6x'+18x=\cos 2t, \quad x(0)=1, \quad x'(0)=-1.$$

Let $\mathscr{L}(x)=X$. Compute (using Fact 15.1 and Table 14.5.1)

$$
\begin{aligned}
\mathscr{L}(x')&=sX-1\\
\mathscr{L}(x'')&=s(sX-1)+1=s^2X-s+1\\
\mathscr{L}(\cos(2t))&=\frac{s}{s^2+4}
\end{aligned}
$$

Apply $\mathscr{L}$ to the original Differential Equation to obtain

$$
\begin{gathered}
(s^2X-s+1)+6(sX-1)+18X=\frac{s}{s^2+4}\\
(s^2+6s+18)X-s-5=\frac{s}{s^2+4}\\
X=\frac{s}{(s^2+4)(s^2+6s+18)}+\frac{s+5}{s^2+6s+18}.
\end{gathered}
$$

We apply the technique of partial fractions to

$$\frac{s}{(s^2+4)(s^2+6s+18)}.$$

Before some one "cleaned this expression up", it looked like

$$\frac{s}{(s^2+4)(s^2+6s+18)} = \frac{A+Bs}{s^2+4} + \frac{C+Ds}{s^2+6s+18}$$

for some constants A, B, C, and D. Multiply both sides by $(s^2+4)(s^2+6s+18)$ to see that

$$s = (A+Bs)(s^2+6s+18)+(C+Ds)(s^2+4).$$

We "equate the corresponding coefficients" in order to find A, B, C and D. (In other words, the left side is a polynomial in the variable s with real number coefficients and the right side is a polynomial in the variable s with real number coefficients. These two polynomials are the same polynomial; so the constant term on the left (which is zero) is equal to the constant term on the right; the coefficient of s on the left (which is 1) is equal to the coefficient of s on the right. At any rate, we "equate the corresponding coefficients" in order to find A, B, C and D.)

$$s = s^3(B+D)+s^2(6B+A+C)+s(18B+6A+4D)+18A+4C$$

Thus,

$$\begin{cases} 0 = B+D \\ 0 = 6B+A+C \\ 1 = 18B+6A+4D \\ 0 = 18A+4C \end{cases}$$

The top equation gives $D=-B$. The bottom equation gives $C=-\frac{9}{2}A$, The middle two equations yield

$$\begin{cases} 0 = 6B+A-\frac{9}{2}A \\ 1 = 18B+6A-4B \end{cases}$$

$$\begin{cases} \frac{7}{2}A = 6B \\ 1 = 18B+6(\frac{2}{7})6B-4B \end{cases}$$

Thus, $1 = (14+\frac{72}{7})B$ or $\frac{7}{170} = B$. It follows that

$$A = 6(\tfrac{2}{7})(\tfrac{7}{170}) = \tfrac{6}{85},$$
$$C = -(\tfrac{9}{2})A = -(\tfrac{9}{2})\tfrac{6}{85} = \tfrac{-54}{170}.$$
$$D = -B = \tfrac{-7}{170}.$$

Thus,

$$\begin{aligned} x &= \mathscr{L}^{-1}(X) \\ &= \mathscr{L}^{-1}\left(\frac{A+Bs}{s^2+4} + \frac{C+Ds}{s^2+6s+18} + \frac{s+5}{s^2+6s+18}\right) \\ &= \frac{1}{170}\mathscr{L}^{-1}\left(\frac{12+7s}{s^2+4} + \frac{-54-7s}{s^2+6s+18} + \frac{170s+5(170)}{s^2+6s+18}\right) \end{aligned}$$

Observe that $s^2+6s+18=(s+3)^2+9$.

$$=\frac{1}{170}\left[6\mathscr{L}^{-1}\left(\frac{2}{s^2+4}\right)+7\mathscr{L}^{-1}\left(\frac{s}{s^2+4}\right)+\mathscr{L}^{-1}\left(\frac{163(s+3)+5(170)-54-(163)3}{(s+3)^2+9}\right)\right]$$

Observe that $5(170)-54-(163)3=307$.

$$=\boxed{\frac{1}{170}\left[6\sin(2t)+7\cos(2t)+163e^{-3t}\cos(3t)+\frac{307}{3}e^{-3t}\sin(3t)\right]}.$$

This is problem 38 in section 7.3 in the textbook.

• Now we look at the Gamma function, $\Gamma(x)$.

a Define $\Gamma(x)$.
b Compute $\Gamma(1)$.
c Find $\Gamma(x+1)$ in terms of $\Gamma(x)$.
d Find $\Gamma(2),\Gamma(3),\Gamma(4),\dots$.
e Find $\mathscr{L}(t^a)$.
f Find $\Gamma(\frac{1}{2})$.

• We do (a). The Gamma function is defined to be

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\quad \text{for } 0<x \tag{15.4.4}$$

• We do (b). Observe that

$$\Gamma(1)=\int_0^\infty e^{-t}t^{1-1}dt=\int_0^\infty e^{-t}dt=\lim_{w\to\infty}-e^{-t}\Big|_0^w=\lim_{w\to\infty}-e^{-w}+1=1.$$

We conclude that $\boxed{\Gamma(1)=1}$.

• We do (c). Observe that

$$\Gamma(x+1)=\int_0^\infty e^{-t}t^x dt.$$

Use Integration by Parts, which is

$$\int udv=uv-\int vdu.$$

Take $u=t^x$ and $dv=e^{-t}dt$. Compute $du=xt^{x-1}dt$ and $v=-e^{-t}$. The integration by parts formula yields

$$\int e^{-t}t^x dt=-t^xe^{-t}+x\int e^{-t}t^{x-1}dt.$$

Thus

$$\begin{aligned}\int_0^\infty e^{-t}t^x dt&=\lim_{w\to\infty}-t^xe^{-t}\Big|_0^w+x\int_0^\infty e^{-t}t^{x-1}dt\\&=\lim_{w\to\infty}-w^xe^{-w}+0^we^{-0}+x\Gamma(x)=x\Gamma(x).\end{aligned}$$

We conclude that $\boxed{\Gamma(x+1)=x\Gamma(x)}$.

• We do (d). Observe that
$$\Gamma(2) = 1\Gamma(1) = 1(1) = 1.$$
Observe that
$$\Gamma(3) = 2\Gamma(2) = 2(1) = 2.$$
Observe that
$$\Gamma(4) = 3\Gamma(3) = 3(2) = 3!.$$
Observe that
$$\Gamma(5) = 4\Gamma(3) = 4(3!) = 4!.$$
We conclude that $\boxed{\Gamma(n+1) = n! \text{ for positive integers } n}$.

• We do (e) Observe that
$$\mathscr{L}(t^a) = \int_0^\infty e^{-st} t^a dt.$$
Let $u = st$. It follows that $du = sdt$. If $t = 0$, then $u = 0$. If t goes to ∞, then u goes to ∞. Thus
$$\mathscr{L}(t^a) = \int_0^\infty e^{-u} (\tfrac{u}{s})^a \tfrac{du}{s} = \tfrac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du = \tfrac{1}{s^{a+1}} \Gamma(a+1)$$
(because $\Gamma(a+1) = \int_0^\infty e^{-t} t^a \, dt$.) We conclude that
$$\boxed{\mathscr{L}(t^a) = \frac{1}{s^{a+1}} \Gamma(a+1).}$$

• We do (f). Observe that
$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2} dt.$$
Let $u = t^{1/2}$. It follows that $du = (1/2)t^{-1/2} dt$. When $t = 0$, then u is also zero. As t goes to infinity, then u also goes to ∞. Thus,
$$\Gamma(\tfrac{1}{2}) = 2 \int_0^\infty e^{-u^2} du. \tag{15.4.5}$$

None of us know an anti-derivative for e^{-u^2}; but we can use trickery to calculate this integral. Possibly, you remember from third semester calculus that it is possible to integrate some definite integrals of the form
$$\iint_{\text{region}} e^{-x^2-y^2} dx\,dy$$
by doing the integral in polar coordinates. We use those tricks here!

We calculate $\int_0^\infty e^{-u^2} du$. Later on we will plug our answer back into (15.4.5) in order to read the value of $\Gamma(\frac{1}{2})$.
$$\begin{aligned} &\int_0^\infty e^{-u^2} du \\ &= \sqrt{\left(\int_0^\infty e^{-u^2} du\right)^2} \end{aligned}$$

$$= \sqrt{\left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)}$$

The expression $\int_0^\infty e^{-y^2} dy$ is a number. I can move that number inside the integral $\int_0^\infty e^{-x^2} dx$ if that makes me happy.

$$= \sqrt{\int_0^\infty e^{-x^2}\left(\int_0^\infty e^{-y^2} dy\right) dx}$$

In the integral $\int_0^\infty e^{-y^2} dy$, y is the variable. The value of x has nothing to do with y. If it amuses me, I can move e^{-x^2}, which is a constant as far as y is concerned, inside the integral $\int_0^\infty e^{-y^2} dy$.

$$= \sqrt{\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dy dx}$$

The most recent integral is the double integral over the first quadrant of $e^{-x^2-y^2}$. None of use can do that integral in rectangular coordinates. But it is easy, if we turn it into an integral in polar coordinates. The first quadrant in polar coordinates is $0 \le r \le \infty$ and $0 \le \theta \le \frac{\pi}{2}$. The sneaky thing about the switch from rectangular coordinates to polar coordinates is that $dxdy$ becomes $r\,dr\,d\theta$

$$= \sqrt{\int_0^{\pi/2} \int_0^\infty re^{-r^2} dr\,d\theta}$$

Do a substitution if you like. Let $u = -r^2$, then $du = -2rdr$. I just did the integral in my head. My answer is right because $\frac{d}{dr}(\frac{-1}{2}e^{-r^2}) = re^{-r^2}$.

$$= \sqrt{\int_0^{\pi/2} \left(\lim_{w\to\infty} \frac{-1}{2} e^{-r^2}\Big|_0^w\right) d\theta}$$

$$= \sqrt{\int_0^{\pi/2} \left(\lim_{w\to\infty} \frac{-1}{2} e^{-w^2} + \frac{1}{2}\right) d\theta}$$

$$= \sqrt{\int_0^{\pi/2} \frac{1}{2} d\theta} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

Now we see from (15.4.5) that

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2} du = 2\tfrac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

We conclude that $\boxed{\Gamma(\frac{1}{2}) = \sqrt{\pi}.}$

Reason for Fact 15.1. Observe that

$$\mathscr{L}(f') = \int_0^\infty e^{-st} f'(t)dt.$$

Apply Integration by Parts ($\int u dv = uv - \int v du$) with $u = e^{-st}$ and $dv = f'(t)dt$. Compute $du = -se^{-st}dt$ and $v = f(t)$. It follows that

$$\begin{aligned}\mathscr{L}(f') &= \lim_{w\to\infty} e^{-st}f(t)\Big|_0^w + s\int_0^\infty e^{-st}f(t)dt \\ &= \lim_{w\to\infty} e^{-sw}f(w) - f(0) + s\mathscr{L}(f) = s\mathscr{L}(f) - f(0).\end{aligned}$$

□

Reason for Fact 15.2. Let $F(s) = \mathscr{L}(f(t))$ and let $g(t) = \int_0^t f(\tau)d\tau$. Apply the Fundamental Theorem of Calculus to see that

$$g'(t) = f(t).$$

Observe that

$$\begin{aligned}F(s) &= \mathscr{L}(f(t)) \\ &= \mathscr{L}(g'(t)) \\ &= s\mathscr{L}(g(t)) - g(0) && \text{by Fact 15.1} \\ &= s\mathscr{L}\left(\int_0^t f(\tau)d\tau\right) - 0 \\ &= s\mathscr{L}\left(\int_0^t \left(\mathscr{L}^{-1}(F(s))\right)\Big|_\tau d\tau\right)\end{aligned}$$

Divide both sides by s. Apply $\mathscr{L}^{-1}$ to each side. Obtain

$$\mathscr{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t \left(\mathscr{L}^{-1}(F(s))\right)\Big|_\tau d\tau.$$

□

Recall **Fact 15.3**. If $\mathscr{L}(f(t)) = F(s)$, then $\mathscr{L}(e^{at}f(t) = F(s-a)$.

Reason.

$$\mathscr{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a).$$

Example. Find $\mathscr{L}^{-1}\left(\frac{1}{s(s+3)}\right)$.

There are two ways to do the problem:

- Use the technique of partial fractions and write
$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$
for some constants A and B and then use Table 14.5.1.
- Apply Fact 15.2 which says
$$\mathscr{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t [\mathscr{L}^{-1}(F(s))]\Big|_\tau d\tau.$$

If one does the **partial fractions** approach one multiplies both sides of

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

by $s(s+3)$ to obtain

$$1 = A(s+3) + Bs.$$

Plug in $s=0$ to obtain $A = \frac{1}{3}$. Plug in $s=-3$ to obtain $B = -\frac{1}{3}$. Thus,

$$\mathscr{L}^{-1}\left(\frac{1}{s(s+3)}\right) = \frac{1}{3}\left(\mathscr{L}^{-1}\left(\frac{1}{s}\right) - \mathscr{L}^{-1}s+3\right) = \frac{1}{3}(1-e^{-3t}).$$

If one uses **Fact 15.2**, then

$$\mathscr{L}^{-1}\left(\frac{\frac{1}{s+3}}{s}\right) = \int_0^t \left[\mathscr{L}^{-1}\left(\frac{1}{s+3}\right)\right]\Big|_\tau d\tau = \int_0^t e^{-3\tau} d\tau = \frac{e^{-3\tau}}{-3}\Big|_0^t$$
$$= \tfrac{-1}{3}(e^{-3t}-1).$$

We conclude that

$$\boxed{\mathscr{L}^{-1}\left(\frac{1}{s(s+3)}\right) = \frac{1}{3}(1-e^{-3t}).}$$

Example. Find $\mathscr{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right)$. Again, one can use the technique of partial fractions (although more care is required when one has a double root) or one can use Fact 15.2 twice.

I will use Fact 15.2 twice. Observe that

$$\begin{aligned}
&\mathscr{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right)\\
&= \mathscr{L}^{-1}\left(\frac{\frac{1}{s(s^2+4)}}{s}\right)\\
&= \int_0^t \left[\mathscr{L}^{-1}\left(\frac{1}{s(s^2+4)}\right)\right]\Big|_\tau d\tau\\
&= \int_0^t \left[\int_0^\tau \left[\mathscr{L}^{-1}\left(\frac{1}{(s^2+4)}\right)\right]\Big|_\theta d\theta\right]\Big|_\tau d\tau
\end{aligned}$$

Recall from Table 14.5.1 that $\left[\mathscr{L}^{-1}\left(\frac{1}{(s^2+4)}\right)\right]\Big|_\theta = \frac{1}{2}\sin(2\theta)$. It follows that

$$\begin{aligned}
&\mathscr{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right)\\
&= \int_0^t \left[\int_0^\tau \frac{1}{2}\sin(2\theta)d\theta\right]\Big|_\tau d\tau\\
&= -\frac{1}{4}\int_0^t \left[\cos(2\theta)\Big|_0^\tau\right] d\tau\\
&= -\frac{1}{4}\int_0^t (\cos(2\tau)-1)d\tau
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4}\left[\frac{\sin(2\tau)}{2} - \tau\right]\Bigg|_0^t \\
&= -\frac{1}{4}\left(\frac{\sin(2t)}{2} - t\right) \\
&= \frac{1}{4}t - \frac{1}{8}\sin(2t).
\end{aligned}$$

We conclude that $\boxed{\mathscr{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \frac{1}{4}t - \frac{1}{8}\sin(2t).}$

Example 15.5. Use Laplace transforms to solve

$$x'' + 8x' + 15x = 0, \quad x(0) = 2, \quad x'(0) = -3.$$

To work: Let $X(s) = \mathscr{L}(x)$. Apply Fact 15.1 to see that

$$\begin{aligned}
\mathscr{L}(x') &= s\mathscr{L}(x) - x(0) = sX - 2 \\
\mathscr{L}(x'') &= s\mathscr{L}(x') - x(0') = s(sX-2) + 3 = s^2X - 2s + 3
\end{aligned}$$

Apply $\mathscr{L}$ to $x'' + 8x' + 15x = 0$ to obtain

$$(s^2X - 2s + 3) + 8(sX - 2) + 15X = 0.$$

$$(s^2 + 8s + 15)X - 2s + 3 - 16 = 0$$

$$X = \frac{13+2s}{s^2+8s+15}.$$

The denominator factors as $(s+3)(s+5)$. Apply the technique of partial fractions. We look for constants A and B with

$$\frac{13+2s}{s^2+8s+15} = \frac{A}{s+3} + \frac{B}{s+5}.$$

Multiply both sides by $(s+3)(s+5)$ to see that

$$13 + 2s = A(s+5) + B(s+3).$$

Plug in $s = -3$ to learn that

$$13 - 6 = A(2)$$

So, $A = \frac{7}{2}$. Plug in $s = -5$ to learn that

$$13 - 10 = -2B$$

So, $B = \frac{-3}{2}$. It is probably a good idea to check that

$$\frac{13+2s}{s^2+8s+15} \text{ really is equal to } \frac{\frac{7}{2}}{s+3} + \frac{\frac{-3}{2}}{s+5}.$$

At any rate,

$$X = \frac{-\frac{3}{2}}{s+5} + \frac{\frac{7}{2}}{s+3}$$

and

$$x = \mathscr{L}^{-1}(X) = -\frac{3}{2}\mathscr{L}^{-1}\left(\frac{1}{s+5}\right) + \frac{7}{2}\mathscr{L}^{-1}\left(\frac{1}{s+3}\right) = -\frac{3}{2}e^{-5t} + \frac{7}{2}e^{-3t}.$$

The solution of the Initial Value Problem is

$$x(t) = -\frac{3}{2}e^{-5t} + \frac{7}{2}e^{-3t}.$$

Check.

$$x(t) = -\frac{3}{2}e^{-5t} + \frac{7}{2}e^{-3t}.$$

$$x'(t) = \frac{15}{2}e^{-5t} - \frac{21}{2}e^{-3t}$$

$$x''(t) = \frac{-75}{2}e^{-5t} + \frac{63}{2}e^{-3t}$$

$$x'' + 8x' + 15x = \left(\frac{-75}{2}e^{-5t} + \frac{63}{2}e^{-3t}\right) + 8\left(\frac{15}{2}e^{-5t} - \frac{21}{2}e^{-3t}\right) + 15\left(-\frac{3}{2}e^{-5t} + \frac{7}{2}e^{-3t}\right)$$

$$= \frac{-75+120-45}{2}e^{-5t} + \frac{63-168+105}{2}e^{-3t} = 0\checkmark$$

$$x(0) = -\frac{3}{2} + \frac{7}{2} = 2\checkmark$$

$$x'(0) = \frac{15}{2} - \frac{21}{2} = -3\checkmark$$

16. Section 7.4: More Laplace transforms.

There is one definition and there are three facts in this section.

Definition 16.1. If $f(t)$ and $g(t)$ are functions, then the convolution of f and g is

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau.$$

Fact 16.2. $\mathscr{L}(f)\cdot\mathscr{L}(g)=\mathscr{L}(f*g)$.

Fact 16.3. $\mathscr{L}(tf)=-\frac{d}{ds}(\mathscr{L}(f))$

Fact 16.4. $\mathscr{L}(\frac{f}{t})=\int_s^\infty[\mathscr{L}(f)]\Big|_\sigma d\sigma$ *provided* $\lim\limits_{t\to0^+}\frac{f}{t}$ *is exists and is finite.*

Example. Find $\mathscr{L}^{-1}\left(\frac{s}{(s^2+9)(s^2+4)}\right)$.

Method 1. We use Fact 16.2 Let $f=\mathscr{L}^{-1}(\frac{s}{(s^2+9)(s^2+4)}$. It follows that

$$\begin{aligned}\mathscr{L}(f)&=\frac{s}{(s^2+9)(s^2+4)}=\frac{s}{(s^2+9)}\cdot\frac{1}{(s^2+4)}=\frac{1}{2}\mathscr{L}(\cos(3t))\cdot\mathscr{L}\sin(2t)\\&=\frac{1}{2}\mathscr{L}(\cos(3t)*\sin(2t)).\end{aligned}$$

Apply $\mathscr{L}^{-1}$ to each side to see that

$$f=\frac{1}{2}\cos(3t)*\sin(2t)=\frac{1}{2}\int_0^t\cos(3\tau)\sin(2(t-\tau))d\tau$$

Recall

$$\begin{aligned}\sin(x+y)&=\sin(x)\cos(y)+\cos(x)\sin(y)\\\sin(x-y)&=\sin(x)\cos(y)-\cos(x)\sin(y)\end{aligned}$$

(If you do not remember these identities, they are quick consequences of the Euler Identity; see the argument of 12.A on page 55.) Add the two formulas (and divide by 2) to see that

$$\frac{\sin(x+y)+\sin(x-y)}{2}=\sin(x)\cos(y).$$

Let $x=2(t-\tau)$ and $y=3\tau$ to see that

$$\begin{aligned}\sin(2(t-\tau))\cos(3\tau)&=\frac{\sin(2(t-\tau)+3\tau)+\sin(2(t-\tau)-3\tau)}{2}\\&=\frac{\sin(2t+\tau)+\sin(2t-5\tau)}{2}\end{aligned}$$

It now follows that

$$\begin{aligned}f&=\frac{1}{2}\int_0^t\cos(3\tau)\sin(2(t-\tau))d\tau\\&=\frac{1}{2}\int_0^t\frac{\sin(2t+\tau)+\sin(2t-5\tau)}{2}d\tau\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}\int_0^t \sin(2t+\tau)+\sin(2t-5\tau)d\tau \\
&= \frac{1}{4}\left[-\cos(2t+\tau)-\frac{\cos(2t-5\tau)}{-5}\right]\Bigg|_0^t \\
&= \frac{1}{4}\left[-\cos(2t+t)-\frac{\cos(2t-5t)}{-5}+\cos(2t)+\frac{\cos(2t)}{-5}\right] \\
&= \frac{1}{4}\left[-\cos(3t)+\frac{\cos(3t)}{5}+\frac{4\cos(2t)}{5}\right] \\
&= \frac{1}{4}\left[-\frac{4\cos(3t)}{5}+\frac{4\cos(2t)}{5}\right] \\
&= \boxed{-\frac{\cos(3t)}{5}+\frac{\cos(2t)}{5}}
\end{aligned}$$

Method 2. Use the technique of Partial Fractions. Find constants A, B, C, and D with

$$\frac{s}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9}+\frac{Cs+D}{s^2+4}.$$

Multiply both sides by $(s^2+9)(s^2+4)$ to obtain

$$s = (As+B)(s^2+4)+(Cs+D)(s^2+9)$$
$$s = s^3(A+C)+s^2(B+D)+s(4A+9C)+4B+9D$$

Solve the system of equations

$$\begin{cases} 0 = A+C \\ 0 = B+D \\ 1 = 4A+9C \\ 0 = 4B+9D \end{cases}$$

simultaneously. Observe that $C=-A$, $B=-D$, $1=4A-9A$, $0=-4D+9D$. Thus, $A=-\frac{1}{5}$ $D=0$, $B=0$, $C=\frac{1}{5}$.

$$\begin{aligned}
\mathscr{L}^{-1}\left(\frac{s}{(s^2+9)(s^2+4)}\right) &= -\frac{1}{5}\mathscr{L}^{-1}\left(\frac{s}{s^2+9}\right)+\frac{1}{5}\mathscr{L}^{-1}\left(\frac{s}{s^2+4}\right) \\
&= \boxed{-\frac{1}{5}\cos(3t)+\frac{1}{5}\cos(2t)}.
\end{aligned}$$

Example. Find $\mathscr{L}(te^{-t}\sin^2 t)$. (This is problem 18 in section 7.4.)

We get to work. Recall from calculus that $\sin^2 t = \frac{1}{2}(1-\cos(2t))$. If you do not remember this formula from calculus, it is not a big deal. Use tricks like the one in 12.A on page 55 to see that $\cos(2t)=\cos^2(t)-\sin^2(t)$. It follows that

$$\cos(2t) = (1-\sin^2(t))-\sin^2(t) = 1-2\sin^2(t). \tag{16.4.1}$$

Rewrite (16.4.1) as $\sin^2 t = \frac{1}{2}(1-\cos(2t))$.

It follows that $\mathscr{L}(te^{-t}\sin^2 t) = \frac{1}{2}\mathscr{L}(te^{-t}(1-\cos(2t)))$. We know $\mathscr{L}(1-\cos(2t))$ from Table 14.5.1 on page 69. So we know $\mathscr{L}(e^{-t}(1-\cos(2t)))$ from 15.3 on page 71. Use Fact 16.3 to finish the calculation.

$$
\begin{aligned}
&\mathscr{L}(te^{-t}\sin^2 t) \\
&= \mathscr{L}\left(te^{-t}\tfrac{1}{2}(1-\cos(2t))\right) \\
&= \tfrac{1}{2}\mathscr{L}\left(t(e^{-t}-e^{-t}\cos(2t))\right) \\
&= -\tfrac{1}{2}\tfrac{d}{ds}\mathscr{L}\left(e^{-t}-e^{-t}\cos(2t)\right) \\
&= -\tfrac{1}{2}\tfrac{d}{ds}\left[\tfrac{1}{s+1}-\tfrac{s+1}{(s+1)^2+4}\right] \\
&= -\tfrac{1}{2}\tfrac{d}{ds}\left[\tfrac{(s+1)^2+4-(s+1)^2}{(s+1)((s+1)^2+4)}\right] \\
&= -\tfrac{1}{2}\tfrac{d}{ds}\left[\tfrac{4}{(s+1)(s^2+2s+5)}\right] \\
&= +(\tfrac{1}{2})4\tfrac{(s+1)(2s+2)+(s^2+2s+5)}{(s+1)^2(s^2+2s+5)^2} \\
&= +2\tfrac{3s^2+6s+7}{(s+1)^2(s^2+2s+5)^2}
\end{aligned}
$$

Example. Find $\mathscr{L}\left(\frac{e^t-e^{-t}}{t}\right)$.

We use Fact 16.4, which says that $\mathscr{L}(\frac{f}{t}) = \int_s^\infty [\mathscr{L}(f)]\Big|_\sigma d\sigma$ provided $\lim\limits_{t\to 0^+}\frac{f}{t}$ exists and is finite.

Notice that

$$\lim_{t\to 0^+}\frac{e^t-e^{-t}}{t} = \lim_{t\to 0^+}\frac{e^t+e^{-t}}{1} = 2$$

by L'Hôpital's rule since $\lim\limits_{t\to 0^+} e^t-e^{-t} = 0$ and $\lim\limits_{t\to 0^+} t = 0$. So, we may apply Fact 16.4.

We compute

$$
\begin{aligned}
&\mathscr{L}\left(\frac{e^t-e^{-t}}{t}\right) \\
&= \int_s^\infty [\mathscr{L}(e^t-e^{-t})]\Big|_\sigma d\sigma \\
&= \int_s^\infty \left(\frac{1}{\sigma-1}-\frac{1}{\sigma+1}\right)d\sigma \\
&= \lim_{w\to\infty}\ln\left|\frac{\sigma-1}{\sigma+1}\right|\Bigg|_s^w \\
&= \lim_{w\to\infty}\ln\left|\frac{w-1}{w+1}\right| - \ln\left|\frac{s-1}{s+1}\right|
\end{aligned}
$$

Recall that $\lim\limits_{w\to\infty}\frac{w-1}{w+1}=1$ and $\ln(1)=0$.

$$\begin{aligned}&= -\ln\left|\frac{s-1}{s+1}\right|\\ &= \ln\left|\frac{s+1}{s-1}\right|\end{aligned}$$

Example. Find $\mathscr{L}^{-1}\ln\left(\frac{s^2+1}{(s+2)(s-3)}\right)$.

My thought is that it would be easy to find $\mathscr{L}^{-1}$ of the derivative of

$$\ln\left(\frac{s^2+1}{(s+2)(s-3)}\right);$$

so lets use Fact 16.3, which is

$$\mathscr{L}(tf)=-\frac{d}{ds}(\mathscr{L}(f)).$$

Let $f=\mathscr{L}^{-1}\ln\left(\frac{s^2+1}{(s+2)(s-3)}\right)$. We want to find f. We first find tf. Indeed,

$$\begin{aligned}\mathscr{L}(tf)&=-\frac{d}{ds}(\mathscr{L}(f))\\ &=-\frac{d}{ds}\ln\left(\frac{s^2+1}{(s+2)(s-3)}\right)\\ &=-\left[\frac{2s}{s^2+1}-\frac{1}{s+2}-\frac{1}{s-3}\right]=\qquad \mathscr{L}(-2\cos(t)+e^{-2t}+e^{3t}).\end{aligned}$$

Apply $\mathscr{L}^{-1}$ to both sides and divide by t in order to conclude that

$$\boxed{f=\frac{-2\cos(t)+e^{-2t}+e^{3t}}{t}.}$$

Example. Find $\mathscr{L}^{-1}\left(\frac{s+1}{(s^2+2s+5)^3}\right)$.

It is not hard to integrate $\frac{s+1}{(s^2+2s+5)^3}$. We would get something like $\frac{1}{s^2+2s+5}^2$. We could probably do $\mathscr{L}^{-1}$ of that. Certainly, the relevant fact is Fact 16.4 $\mathscr{L}(\frac{f}{t})=\int_s^\infty[\mathscr{L}(f)]\Big|_\sigma d\sigma$ provided $\lim\limits_{t\to 0^+}\frac{f}{t}$ is exists and is finite.

I propose that we let $f=\mathscr{L}^{-1}\left(\frac{s+1}{(s^2+2s+5)^3}\right)$. Keep in mind that we want to find f. We start by finding $\frac{f}{t}$. Of course, we know $\mathscr{L}(f)$ even though we do not yet know f. We get to work:

$$\begin{aligned}\mathscr{L}(\frac{f}{t})&=\int_s^\infty[\mathscr{L}(f)]\Big|_\sigma d\sigma\\ &=\int_s^\infty\frac{\sigma+1}{(\sigma^2+2\sigma+5)^3}d\sigma\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{4} \lim_{w\to\infty} \frac{1}{(\sigma^2+2\sigma+5)^2}\Big|_s^w \\
&= \frac{1}{4}\frac{1}{(s^2+2s+5)^2} \\
&= \frac{1}{4}\frac{1}{((s+1)^2+4)^2}
\end{aligned}
$$

Recall from (15.3.1) on page 72 that $\mathscr{L}^{-1}\left(\frac{1}{(s^2+k^2)^2}\right) = \frac{1}{2k^3}(\sin(kt) - kt\cos(kt))$. Also use Fact 15.3 on page 71.

$$= \mathscr{L}\Big(\frac{1}{4}e^{-t}\frac{1}{2(2^3)}(\sin(2t)-2t\cos(2t))\Big)$$

Take $\mathscr{L}^{-1}$ of both sides of the equation to obtain

$$\frac{f}{t} = \frac{1}{4}e^{-t}\frac{1}{2(2^3)}(\sin(2t)-2t\cos(2t))$$

(By the way, now that we know $\frac{f}{t}$, it is clear $\lim\limits_{t\to 0^+}\frac{f}{t}$ exists and is finite. So our calculation is legal.) We conclude that

$$\boxed{f = \frac{1}{64}te^{-t}(\sin(2t)-2t\cos(2t)).}$$

Example. Find a non-trivial solution of

$$tx'' - (4t+1)x' + 4tx + 2x = 0, \quad x(0) = 0.$$

The function $x(t) = 0$ for all t is a solution of this homogeneous linear Differential Equation. The instruction tells us to find a non-trivial solution; that is, we are supposed to find another solution in addition to the trivial solution $x(t) = 0$ for all t.

We use the method of Laplace transforms. Let $X = \mathscr{L}(x)$. We compute

$$
\begin{aligned}
\mathscr{L}(x') &= sX \\
\mathscr{L}(x'') &= s^2X - x'(0) \\
\mathscr{L}(tx) &= -\frac{d}{ds}X = -X' \\
\mathscr{L}(tx') &= -\frac{d}{ds}(sX) = -(sX'+X)
\end{aligned}
$$

$$\mathscr{L}(tx'') = -\frac{d}{ds}(s^2X - x'(0)) = -(s^2X'+2sX)$$

Apply $\mathscr{L}$ to the original Differential Equation to obtain

$$-(s^2X'+2sX) + 4(sX'+X) - sX - 4X' + 2X = 0$$

The most recent equation is a Differential Equation for $X(s)$. The new DE is a first order problem. Maybe it is easier than the original second order problem. Rewrite the new problem as

$$(-s^2+4s-4)X'+(-2s+4-s+2)X=0$$
$$(-s^2+4s-4)X'+(-3s+6)X=0$$
$$-(s-2)^2X'-3(s-2)X=0$$

Divide both sides of the equation by $-(s-2)$.

$$(s-2)X'+3X=0$$

We can separate the variables.

$$(s-2)\frac{dX}{ds}=-3X$$
$$\int\frac{dX}{X}=\int\frac{-3}{s-2}ds$$
$$\ln|X|=-3\ln|s-2|+C$$

Exponentiate to obtain

$$|X|=e^C|s-2|^{-3}$$
$$X=\pm e^C(s-2)^{-3}$$

Let $K=\pm e^C$. Recall that

$$\mathscr{L}(t^2)=\frac{2!}{s^3}\quad\text{and}\quad\mathscr{L}(e^{at}f(t))=[\mathscr{L}(f(t))]|_{s-a}.$$

It follows that $\mathscr{L}(e^{2t}t^2)=\frac{2}{(s-2)^3}$ and

$$x=\mathscr{L}^{-1}(X)=\mathscr{L}^{-1}(K(s-2)^{-3})=\tfrac{K}{2}e^{2t}t^2=Be^{2t}t^2,$$

where B is the constant $\frac{K}{2}$. We conclude that $\boxed{x=Be^{2t}t^2}$ is a solution for any constant B.

Check. It is clear that if $x=e^{2t}t^2$ is a solution of the Differential Equation, then $x=Be^{2t}t^2$ is a solution for any constant B. So, it is good enough to check that $x=e^{2t}t^2$ is a solution of the Differential Equation. We calculate

$$x'=2e^{2t}t+2e^{2t}t^2$$

$$\begin{aligned}x''&=\left(2e^{2t}+4e^{2t}t\right)+\left(4e^{2t}t+4e^{2t}t^2\right)\\&=2e^{2t}+8e^{2t}t+4e^{2t}t^2\end{aligned}$$

Plug x into the left side of the DE to obtain

$$tx''-(4t+1)x'+4tx+2x=\left\{\begin{array}{l}\quad 2e^{2t}t+8e^{2t}t^2+4e^{2t}t^3\\ \qquad\qquad -8e^{2t}t-8e^{2t}t^2\\ -2e^{2t}t-2e^{2t}t^2\\ \qquad\qquad +2e^{2t}t^2+4e^{2t}t^3\end{array}\right\}=0.\checkmark$$

The reason for Fact 16.2. We show that

$$\mathscr{L}(f)\cdot\mathscr{L}(g)=\mathscr{L}(f*g).$$

The first thing I will do is a little technical, but it isn't hard. Indeed, I don't really have to do it, but if I didn't do it, I would have to work a little harder at a later point.

Notice that

$$\mathscr{L}(f)=\int_0^\infty e^{-st}f(t)dt,\quad \mathscr{L}(g)=\int_0^\infty e^{-st}g(t)dt,\quad \text{and}$$

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

all are about evaluating f and g at non-negative numbers only. In particular, if we change the value of f or g at some negative number, then we will not have changed $\mathscr{L}(f)$, $\mathscr{L}(g)$, $f*g$, or $\mathscr{L}(f*g)$. So let

$$f^+(t)=\begin{cases} f(t) & \text{for } 0\le t\\ 0 & \text{for } t<0\end{cases}\quad \text{and}\quad g^+(t)=\begin{cases} g(t) & \text{for } 0\le t\\ 0 & \text{for } t<0.\end{cases}$$

The point is that

$$\mathscr{L}(f^+)=\mathscr{L}(f),\quad \mathscr{L}(g^+)=\mathscr{L}(g),\quad \text{and}\quad \mathscr{L}(f*g)=\mathscr{L}(f^+*g^+).$$

We compute

$$\begin{aligned}\mathscr{L}(f*g)&=\mathscr{L}(f^+*g^+)\\ &=\int_0^\infty e^{-st}(f^+*g^+)(t)dt\\ &=\int_0^\infty e^{-st}\left(\int_0^t (f^+(\tau)g^+(t-\tau)d\tau\right)dt\end{aligned}$$

The basic trick is to let $u=t-\tau$. When we do this e^{-st} becomes $e^{-s(u+\tau)}=e^{-su}e^{-s\tau}$. We then "pull the integrals apart" using techniques from Math 241.

If $t<\tau<\infty$, then $t-\tau<0$, $g^+(t-\tau)=0$, and $\int_t^\infty f^+(\tau)g^+(t-\tau)d\tau=0$.

$$=\int_0^\infty e^{-st}\left(\int_0^\infty f^+(\tau)g^+(t-\tau)d\tau\right)dt$$

As far as the inner integral is concerned e^{-st} is a constant; so we can move e^{-st} inside the inner integral. The parentheses no longer have any significance.

$$\begin{aligned}&=\int_0^\infty \left(\int_0^\infty e^{-st}f^+(\tau)g^+(t-\tau)d\tau\right)dt\\ &=\int_0^\infty \int_0^\infty e^{-st}f^+(\tau)g^+(t-\tau)\,d\tau\,dt\end{aligned}$$

This object is called a double integral. One computes a double integral be using iterated integrals. The given iterated integral is taken over the whole first quadrant in the (τ,t)-plane. One visualizes this iterated integral as saying for each fixed t (with $0 \le t < \infty$), τ goes from 0 to ∞. (See the picture on the left on the next page.) One can also fill this region using "vertical lines". In this second approach, one says, "For each fixed τ (with $0 \le \tau < \infty$), t goes from 0 to ∞." This is the picture on the right on the next page.

$$= \int_0^\infty \int_0^\infty e^{-st} f^+(\tau) g^+(t-\tau)\, dt\, d\tau$$

Now we make the promised substitution. We let $u = t - \tau$ in the inner integral. Remember τ is constant in the inner integral! Observe that $du = dt$. When $t = 0$, $u = -\tau$. As t goes to ∞, then u also goes to ∞.

$$= \int_0^\infty \int_{-\tau}^\infty e^{-s(u+\tau)} f^+(\tau) g^+(u)\, du\, d\tau$$

Of course, $g^+(u) = 0$ for $-\tau \le u < 0$; so $\int_{-\tau}^0 e^{-s(u+\tau)} f^+(\tau) g^+(u)\, du = 0$ and $\int_{-\tau}^\infty e^{-s(u+\tau)} f^+(\tau) g^+(u)\, du = \int_0^\infty e^{-s(u+\tau)} f^+(\tau) g^+(u)\, du$.

$$= \int_0^\infty \int_0^\infty e^{-s(u+\tau)} f^+(\tau) g^+(u)\, du\, d\tau$$

We can separate the exponential $e^{-s(u+\tau)} = e^{-su} e^{-s\tau}$

$$= \int_0^\infty \int_0^\infty e^{-su} e^{-s\tau} f^+(\tau) g^+(u)\, du\, d\tau$$

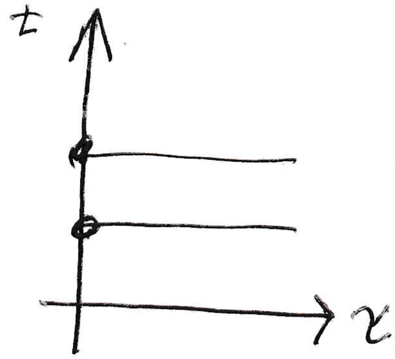
As far as the inner integral is concerned, $e^{-s\tau} f^+(\tau)$ is a constant, because it does not involve u.

$$= \int_0^\infty e^{-s\tau} f^+(\tau) \int_0^\infty e^{-su} g^+(u)\, du\, d\tau$$

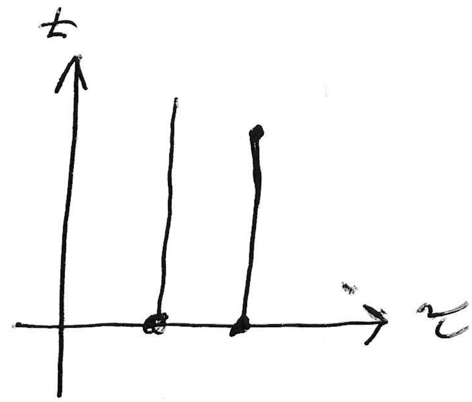
The expression $\int_0^\infty e^{-su} g^+(u)\, du$ does not involve the variable τ of the outer integral. We may pull it out!

$$\begin{aligned} &= \int_0^\infty e^{-s\tau} f^+(\tau) d\tau \int_0^\infty e^{-su} g^+(u)\, du \\ &= \mathscr{L}(f^+) \cdot \mathscr{L}(g^+) = \mathscr{L}(f) \cdot \mathscr{L}(g). \end{aligned}$$

$\square$



$$\int_0^\infty \int_0^\infty h(\tau, t)\, d\tau\, dt$$



$$\int_0^\infty \int_0^\infty h(\tau, t)\, dt\, d\tau$$

These two integrals are equal.

Reason for Fact 16.3. We show what

$$\mathscr{L}(tf) = -\frac{d}{ds}\mathscr{L}(f)$$

is all about.

Observe that

$$\begin{aligned}\frac{d}{ds}\mathscr{L}(f) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt\\ &= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}f(t))dt\end{aligned}$$

This is the only interesting step. It is called the Leibniz integral. One learns it in a graduate course in Mathematics, Math 703.

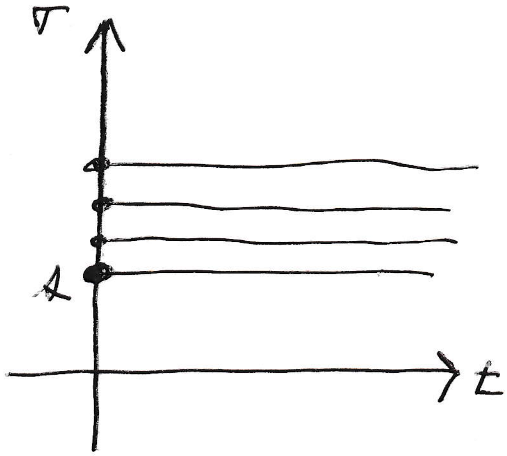
$$\begin{aligned} &= \int_0^\infty -t(e^{-st}f(t))dt\\ &= -\int_0^\infty e^{-st}tf(t)dt\\ &= -\mathscr{L}(tf(t)).\end{aligned}$$

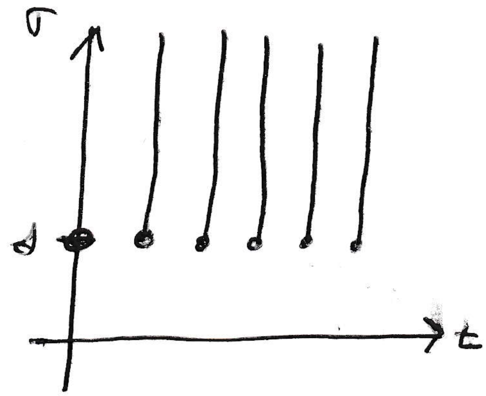
□

Reason for Fact 16.4. We show that $\mathscr{L}(\frac{f}{t}) = \int_s^\infty [\mathscr{L}(f)]\Big|_\sigma d\sigma$, provided $\lim\limits_{t\to 0^+}\frac{f}{t}$ exists and is finite.

Observe that

$$\begin{aligned} &\int_s^\infty [\mathscr{L}(f)]\Big|_\sigma d\sigma\\ &= \int_s^\infty \int_0^\infty e^{-\sigma t} f(t)\,dt\,d\sigma\\ &= \int_0^\infty \int_s^\infty e^{-\sigma t} f(t)\,d\sigma\,dt && \text{See the picture on the next page.}\\ &= \int_0^\infty \lim_{w\to\infty} \tfrac{-1}{t} e^{-\sigma t} f(t)\Big|_s^w dt\\ &= \int_0^\infty \left(\lim_{w\to\infty} \tfrac{-1}{t}e^{-wt}f(t) - \tfrac{-1}{t}e^{-st}f(t)\right) dt && \text{The limit is obviously zero.}\\ &= \int_0^\infty e^{-st}\tfrac{f(t)}{t}\,dt\\ &= \mathscr{L}(\tfrac{f(t)}{t}), && \text{provided } \lim_{t\to 0^+}\tfrac{f(t)}{t} \text{ exists and is finite.}\end{aligned}$$





$$\int_{s}^{\infty}\int_{0}^{\infty} h(t,\tau)\,dt\,d\tau \qquad \int_{0}^{\infty}\int_{s}^{\infty} h(t,\tau)\,d\tau\,dt$$

These integrals are equal.

Fact 16.5. $f * g = g * f$

Reason

$$(g * f)(t) = \int_0^t g(\tau) f(t-\tau) d\tau$$

Let $u = t - \tau$. Compute $du = -d\tau$. When $\tau = 0$, then $u = t$. When $\tau = t$, then $u = 0$.

$$\begin{aligned} &= -\int_t^0 g(t-u) f(u) du \\ &= \int_0^t g(t-u) f(u) du \\ &= (f * g)(t). \end{aligned}$$

□