Math 242, Final Exam, Spring, 2024

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 100 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Solve the Initial Value problem**

$$\tfrac{dx}{dt} = x - 5, \quad x(0) = x_0.$$

Graph the solution of the Initial Value Problem for a few different choices of x_0.

Separarte the variables and integrate:

$$\int \frac{dx}{x-5} = \int dt$$

$$\ln|x-5| = t + C$$

Exponentiate:

$$e^{\ln|x-5|} = e^{t+C}$$

$$|x-5| = e^C e^t$$

$$x - 5 = \pm e^C e^t$$

Let K denote the constant $\pm e^C$.

$$x = 5 + Ke^t$$

Use the Initial Condition $x(0) = x_0$ to evaluate K:

$$x_0 = x(0) = 5 + Ke^0$$

So $x_0 - 5 = K$. The solution of the Initial Value Problem is

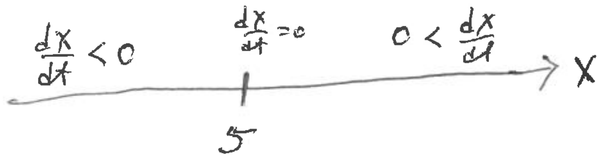
$$\boxed{x = 5 + (x_0 - 5)e^t}$$

Check. We compute
$$\frac{dx}{dt} = (x_0 - 5)e^t.$$
We also compute
$$x - 5 = (5 + (x_0 - 5)e^t) - 5 = (x_0 - 5)e^t.$$
Thus, $\frac{dx}{dt} = x - 5$✓. Also, $x(0) = 5 + (x_0 - 5)e^0 = x_0$✓.

The picture is on the next page.

The graph of the solution of the Initial Value ...

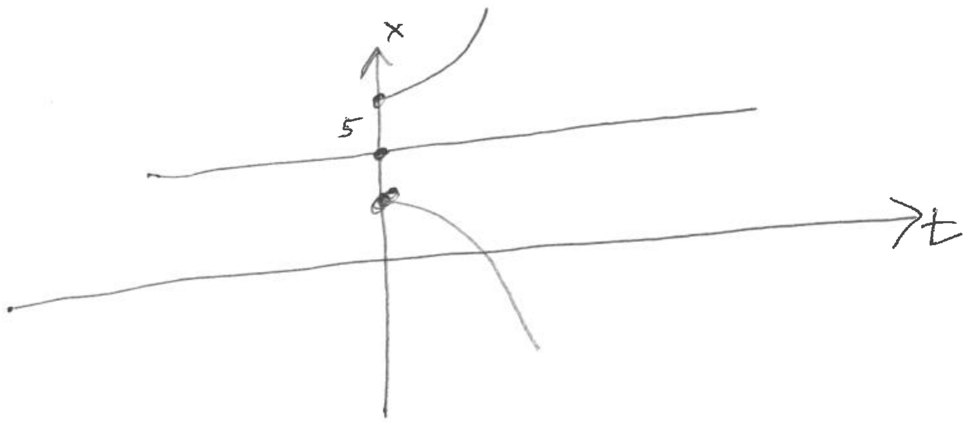
Problem $\frac{dx}{dt} = x - 5$, $x(0) = x_0$ for a few choices of x_0



If $x_0 = 5$, then $x(t) = 5$ for all t

If $5 < x_0$, then $x = x(t)$ increases away from 5 for all $0 \le t$

If $x_0 < 5$, then $x = x(t)$ decreases away from 5 for all $0 \le t$.



(2) **Find $\mathcal{L}^{-1}\left(\frac{s+8}{s^2+4s+13}\right)$.**

$$\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{s+8}{s^2+4s+13}\right)\\
&=\mathcal{L}^{-1}\left(\frac{s+8}{(s^2+4s+4)+9}\right)\\
&=\mathcal{L}^{-1}\left(\frac{(s+2)+6}{(s+2)^2+9}\right)\\
&=\mathcal{L}^{-1}\left(\frac{(s+2)}{(s+2)^2+9}\right)+2\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right)
\end{aligned}$$

Recall that $\mathcal{L}^{-1}(\frac{s}{s^2+9}) = \cos 3t$; it follows that $\mathcal{L}^{-1}\left(\frac{(s+2)}{(s+2)^2+9}\right) = e^{-2t}\cos 3t$. In a similar manner $\mathcal{L}^{-1}(\frac{3}{s^2+9}) = \sin 3t$ and $\mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right) = e^{-2t}\sin 3t$

$$= \boxed{e^{-2t}\cos 3t + 2e^{-2t}\sin 3t.}$$

(3) **Use the method of Laplace transforms to solve the Initial Value Problem**

$$x'' - x' - 6x = 0, \quad x(0) = 2, \quad \textbf{and} \quad x'(0) = -1.$$

Let $X = \mathcal{L}(x)$. Use $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ twice to calculate

$$\begin{aligned}
\mathcal{L}(x') &= s\mathcal{L}(x) - x(0) = sX - 2\\
\mathcal{L}(x'') &= s\mathcal{L}(x') - x'(0) = s^2X - 2s + 1
\end{aligned}$$

Apply $\mathcal{L}$ to the Initial Value Problem to obtain

$$s^2X - 2s + 1 - (sX - 2) - 6X = 0$$

$$(s^2 - s - 6)X = 2s - 3$$

$$X = \frac{2s-3}{(s-3)(s+2)}. \tag{1}$$

We use the technique of Partial Fractions and find numbers A and B with

$$\frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

Multiply both sides by $(s-3)(s+2)$:

$$2s - 3 = A(s+2) + B(s-3)$$

Plug $s = -2$ into this equation to learn $B = \frac{7}{5}$. Plug $s = 3$ into the equation to learn $A = \frac{3}{5}$. We have calculated that

$$\frac{2s-3}{(s-3)(s+2)} = \frac{1}{5}\left[\frac{3}{s-3} + \frac{7}{s+2}\right].$$

Resume from (1):

$$x = \mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{2s-3}{(s-3)(s+2)}\right) = \frac{1}{5}\left[3\mathcal{L}^{-1}(\frac{1}{s-3}) + 7\mathcal{L}^{-1}(\frac{1}{s+2})\right]$$

$$\boxed{x = \tfrac{1}{5}(3e^{3t} + 7e^{-2t})}$$

Check. Plug

$$\begin{aligned} x &= \tfrac{1}{5}(3e^{3t} + 7e^{-2t}) \\ x' &= \tfrac{1}{5}(9e^{3t} - 14e^{-2t}) \\ x'' &= \tfrac{1}{5}(27e^{3t} + 28e^{-2t}) \end{aligned}$$

into $x'' - x' - 6x$ and obtain

$$\frac{1}{15}\left\{\begin{array}{l} (27e^{3t} + 28e^{-2t}) \\ -(9e^{3t} - 14e^{-2t}) \\ -6(3e^{3t} + 7e^{-2t}) \end{array}\right\} = 0\checkmark;$$

$x(0) = \frac{1}{5}(3+7) = 2\checkmark$; $x'(0) = \frac{1}{5}(9-14) = -1\checkmark$. The proposed answer does everything it is supposed to do. It is correct.

(4) **Find the general solution of $xy' + 3y = 2x^5$. (In this problem $y = y(x)$.)**

This is a First Order Linear problem. Divide both sides by x to put the equation in the form $y' + P(x)y = Q(x)$:

$$y' + \tfrac{3}{x}y = 2x^4.$$

Multiply both sides by

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = x^3 :$$

$$x^3y' + 3x^2y = 2x^7.$$

Observe that the left side is equal to $\frac{d}{dx}(x^3y)$. Integrate both sides with respect to x:

$$x^3y = \tfrac{x^8}{4} + C$$

$$\boxed{y = \frac{x^5}{4} + Cx^{-3}}.$$

Check. We compute $\frac{dy}{dx} = 5\frac{x^4}{4} - 3Cx^{-4}$. It follows that

$$xy' + 3y = x(5\tfrac{x^4}{4} - 3Cx^{-4}) + 3(\tfrac{x^5}{4} + Cx^{-3}) = 2x^5.\checkmark$$

(5) **Find the general solution of** $xyy' = x^2 + 3y^2$**. (In this problem** $y = y(x)$**.)**

This equation is homogeneous of degree two in x and y. Divide both sides by x^2:

$$\tfrac{y}{x}y' = 1 + 3(\tfrac{y}{x})^2.$$

Let $v = \frac{y}{x}$. It follows that $xv = y$. Take $\frac{d}{dx}$ of both sides to see that

$$x\tfrac{dv}{dx} + v = \tfrac{dy}{dx}.$$

The original Differential Equation has become

$$v(v + x\tfrac{dv}{dx}) = 1 + 3v^2$$

$$v^2 + xv\tfrac{dv}{dx} = 1 + 3v^2$$

$$xv\tfrac{dv}{dx} = 1 + 2v^2$$

$$\int \tfrac{v}{1+2v^2}dv = \int \tfrac{dx}{x}$$

$$\tfrac{1}{4}\ln(1 + 2v^2) = \ln|x| + C$$

(The expression $1 + 2v^2$ is guaranteed to be positive.)

$$\ln(1 + 2v^2) = 4\ln|x| + 4C$$

Exponentiate

$$1 + 2v^2 = e^{4c}e^{4\ln|x|}$$

$$1 + 2v^2 = e^{4c}x^4$$

(The expression $|x|^4$ is equal to x^4.) We may as well let K be the constant e^{4C}.

$$2v^2 = Kx^4 - 1$$

$$2(\tfrac{y}{x})^2 = Kx^4 - 1$$

$$2y^2 = Kx^6 - x^2$$

$$\boxed{y = \pm\tfrac{1}{\sqrt{2}}\sqrt{Kx^6 - x^2}.}$$

Check. We compute

$$\begin{aligned} xyy' &= x\Big(\pm\frac{1}{\sqrt{2}}\sqrt{Kx^6-x^2}\Big)\Big(\pm\frac{1}{\sqrt{2}}\frac{6Kx^5-2x}{2\sqrt{Kx^6-x^2}}\Big) \\ &= \frac{1}{4}x\Big(\sqrt{Kx^6-x^2}\Big)\Big(\frac{6Kx^5-2x}{\sqrt{Kx^6-x^2}}\Big) \\ &= \tfrac{1}{4}x(6Kx^5-2x) \\ &= \tfrac{1}{4}(6Kx^6-2x^2) \\ &= \tfrac{1}{2}(3Kx^6-x^2) \end{aligned}$$

On the other hand,

$$\begin{aligned} x^2+3y^2 &= x^2+3\Big(\pm\frac{1}{\sqrt{2}}\sqrt{Kx^6-x^2}\Big)^2 \\ &= x^2+3(\tfrac{1}{2})(Kx^6-x^2). \end{aligned}$$

Our proposed answer does satisfy the Differential Equation. It is correct.

(6) **Find the general solution of $y''-4y'+13y=0$. (In this problem $y=y(x)$.)**

This is a homogeneous second order Linear Differential Equation. We try $y=e^{rx}$. We consider the characteristic equation

$$r^2-4r+13=0.$$

We use the quadratic formula: if $ar^2+br+c=0$, then $r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. For us, $a=1$, $b=-4$, and $c=13$; so

$$r=\frac{4\pm\sqrt{16-52}}{2}=\frac{4\pm\sqrt{-36}}{2}=\frac{4\pm 6i}{2}=2\pm 3i.$$

Thus, the general solution of the Differential Equation is

$$\boxed{y=e^{2x}(c_1\cos 3x+c_2\sin 3x).}$$

Check. Plug

$$\begin{aligned} y &= e^{2x}(c_1\cos 3x+c_2\sin 3x) \\ y' &= e^{2x}(-3c_1\sin 3x+3c_2\cos 3x)+2e^{2x}(c_1\cos 3x+c_2\sin 3x) \\ &= e^{2x}(-1c_1\sin 3x+5c_2\cos 3x) \\ y'' &= e^{2x}(-3c_1\cos 3x-15c_2\sin 3x)+2e^{2x}(-1c_1\sin 3x+5c_2\cos 3x) \\ &= e^{2x}(7c_1\cos 3x-17c_2\sin 3x) \end{aligned}$$

into $y'' - 4y' + 13y$ and obtain

$$\begin{cases} e^{2x}(7c_1 \cos 3x - 17c_2 \sin 3x) \\ -4e^{2x}(-1c_1 \sin 3x + 5c_2 \cos 3x) \\ +13e^{2x}(c_1 \cos 3x + c_2 \sin 3x) \end{cases}$$

$$= e^{2x}\Big((7 - 20 + 13)\cos 3x + (-17 + 4 + 13)\sin 3x\Big) = 0.\checkmark$$

(7) **Newton's Law of Cooling states that the rate at which an object cools is proportional to difference between the temperature of the object and the temperature of the surrounding medium. At time zero a roast, with a temperature of 375 degrees F, is taken from the oven and placed in a room with temperature 70 degrees F. Twenty five minutes later, the temperature of the roast is 225 degrees F. When will the temperature of the roast reach 125 degrees?**

Let $T(t)$ be the temperature of the roast (in degrees F) at time t (in minutes). We are told that

$$\begin{aligned} \frac{dT}{dt} &= k(T - 70) \\ T(0) &= 375 \\ T(25) &= 225 \end{aligned}$$

We are supposed to find t with $T(t) = 125$.

We separate the variables and integrate:

$$\int \frac{dT}{T - 70} = \int k\, dt$$

$$\ln(T - 70) = kt + C$$

(In this problem $T - 70$ is positive; so $|T - 70| = T - 70$.)
Exponentiate:

$$T - 70 = e^C e^{kt}.$$

Use $T(0) = 375$ to evaluate e^C:

$$375 - 70 = e^C(1);$$

so $305 = e^C$ and

$$T - 70 = 305e^{kt}.$$

Use $T(25) = 225$ to evaluate k:

$$225 - 70 = 305e^{k(25)}$$

$$\frac{155}{305} = e^{k(25)}$$
$$\ln\left(\tfrac{155}{305}\right) = k(25)$$
$$\tfrac{1}{25}\ln(\tfrac{155}{305}) = k. \tag{2}$$

The temperature of the roast at time t is

$$T(t) = 70 + 305e^{kt} \text{ for } k \text{ given in (2).}$$

Now we find t with $T(t) = 125$:

$$125 = 70 + 305e^{kt}$$
$$\tfrac{55}{305} = e^{kt}$$
$$\ln(\tfrac{55}{305}) = kt$$

The temperature of the roast reaches 125 degrees after $\dfrac{25\ln(\frac{55}{305})}{\ln(\frac{155}{305})}$ minutes.

(8) **Find a particular solution of $y'' - 3y' + 2y = e^x$. In this problem $y = y(x)$.**

The function $y = e^x$ is a solution of the homogeneous equation $y'' - 3y' + 2y = 0$; so it makes no sense to look for a solution of $y'' - 3y' + 2y = e^x$ of the form $y = Ae^x$. Instead we try $y = Axe^x$. We look for a number A with $y = Axe^x$ a solution of $y'' - 3y' + 2y = e^x$.

Plug

$$y = Axe^x$$
$$y' = A(xe^x + e^x)$$
$$y'' = A(xe^x + 2e^x)$$

into $y'' - 3y' + 2y = e^x$ and obtain

$$A([xe^x + 2e^x] - 3[xe^x + e^x] + 2xe^x = e^x$$
$$A([1 - 3 + 2]xe^x + [2 - 3]e^x = e^x$$
$$-Ae^x = e^x$$

Take $A = -1$. Thus,

$y = -xe^x$ is a solution of $y'' - 3y' + 2y = e^x$.

Check. Plug

$$y = -xe^x$$
$$y' = -xe^x - e^x$$
$$y'' = -xe^x - e^x - e^x = -xe^x - 2e^x$$

into $y'' - 3y' + 2y$ and obtaon

$$(-xe^x - 2e^x) - 3(-xe^x - e^x) + 2(-xe^x) = xe^x(-1+3-2) + e^x(-2+3) = e^x. \checkmark.$$

(9) **Find a particular solution of $y'' + 3y' + 2y = x^2$. In this problem $y = y(x)$.**

We look for a solution of the form $y = Ax^2 + Bx + C$ for numbers A, B, and C.

Plug

$$\begin{aligned} y &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

into $y'' + 3y' + 2y = x^2$ and obtain

$$2A + 3[2Ax + B] + 2[Ax^2 + Bx + C] = x^2$$

$$2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$$

So, we want

$$2A = 1, \quad 6A + 2B = 0, \quad 2A + 3B + 2C = 0$$

We take $A = \frac{1}{2}$, $B = -\frac{3}{2}$, and $C = \frac{7}{4}$.

$$\boxed{y = \tfrac{1}{4}(2x^2 - 6x + 7) \text{ is a solution of } y'' + 3y' + 2y = x^2.}$$

Check. Plug

$$\begin{aligned} y &= \tfrac{1}{4}(2x^2 - 6x + 7) \\ y' &= \tfrac{1}{4}(4x - 6) \\ y'' &= \tfrac{1}{4}(4) \end{aligned}$$

into $y'' + 3y' + 2y$ and obtain

$$\tfrac{1}{4}(4+3[4x-6]+2[2x^2-6x+7]) = \tfrac{1}{4}([4-18+14]+[12-12]x+4x^2) = x^2 \checkmark.$$

(10) **Set up, but do not solve, an Initial Value Problem. A 150-gallon tank initially contains 110 pounds of salt dissolved in 80 gallons of water. Brine containing 4 pounds per gallon of salt flows into the tank at the rate of 5 gallons per minute, and the well-stirred mixture flows out of the tank at the rate of 2 gallons per minute. How much salt is in the tank after t minutes?**

Let $x(t)$ be the number of pounds of salt in the tank at time t minutes. We are told that $x(0) = 110$ and that salt enters the tank at the rate

$$4\frac{\text{lb}}{\text{gal}}5\frac{\text{gal}}{\text{min}} = 20\frac{\text{lb}}{\text{min}}$$

and salt leaves the tank at the rate

$$\frac{x(t) \text{ lb}}{80+3t \text{ gal}} 2 \frac{\text{gal}}{\text{min}} = \frac{2x}{80+3t} \frac{\text{lb}}{\text{min}}.$$

Thus x is the solution of the Initial Value Problem:

$$\boxed{\frac{dx}{dt} = 20 - \frac{2x}{80+3t}, \quad x(0) = 110.}$$