Math 242, Exam 2, Spring, 2024

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Is $y = e^x \cos x$ a solution of $y'' - 2y' + 2y = 0$? Explain.**

We compute

$$\begin{aligned} y' &= -e^x \sin x + e^x \cos x \\ y'' &= (-e^x \cos x - e^x \sin x) + (-e^x \sin x + e^x \cos x) \\ &= -2e^x \sin x \end{aligned}$$

Plug y, y', and y'' into $y'' - 2y' + 2y$ to obtain

$$\begin{aligned} &(-2e^x \sin x) - 2(-e^x \sin x + e^x \cos x) + 2e^x \cos x \\ &= (-2e^x \sin x + 2e^x \sin x) + (-2e^x \cos x + 2e^x \cos x) \\ &= 0 + 0 = 0. \end{aligned}$$

$$\boxed{\text{Thus, } y = e^x \cos x \text{ IS a solution of } y'' - 2y' + 2y = 0.}$$

(2) **Solve the Differential Equation**

$$xy\tfrac{dy}{dx} = y^2 + x\sqrt{4x^2 + y^2}.$$

Please check your answer.

This is a homogeneous differential equation: every terms acts like it is homogeneous of degree 2 in x and y. Divide both sides by x^2 to obtain:

$$\tfrac{y}{x}\tfrac{dy}{dx} = (\tfrac{y}{x})^2 + \tfrac{\sqrt{4x^2+y^2}}{x}$$

$$\tfrac{y}{x}\tfrac{dy}{dx} = (\tfrac{y}{x})^2 + \sqrt{\tfrac{4x^2+y^2}{x^2}}$$

$$\tfrac{y}{x}\tfrac{dy}{dx} = (\tfrac{y}{x})^2 + \sqrt{4 + (\tfrac{y}{x})^2}$$

Let $v = \frac{y}{x}$. It follows that $xv = y$ and $x\frac{dv}{dx} + v = \frac{dy}{dx}$. The Differential Equation becomes

$$v(x\tfrac{dv}{dx} + v) = v^2 + \sqrt{4+v^2}$$

Divide both sides by v and subtract v from both sides:

$$x\tfrac{dv}{dx} + v = v + \tfrac{\sqrt{4+v^2}}{v}$$
$$x\tfrac{dv}{dx} = \tfrac{\sqrt{4+v^2}}{v}$$

Multiply both sides by $\frac{v}{\sqrt{4+v^2}}\frac{1}{x}dx$ and integrate

$$\int \tfrac{v}{\sqrt{4+v^2}}dv = \int \tfrac{1}{x}dx$$

Make a substitution; let $u = 4 + v^2$. It follows that $du = 2vdv$.

$$\sqrt{4+v^2} = \ln|x| + C$$

Square both sides; subtract 4 from both sides; take the square root of both sides:

$$\begin{aligned}4 + v^2 &= (\ln|x| + C)^2\\ v^2 &= (\ln|x| + C)^2 - 4\\ v &= \pm\sqrt{(\ln|x| + C)^2 - 4}\\ \tfrac{y}{x} &= \pm\sqrt{(\ln|x| + C)^2 - 4}\end{aligned}$$
$$\boxed{y = \pm x\sqrt{(\ln|x| + C)^2 - 4}}$$

Check We check $y = x\sqrt{(\ln|x| + C)^2 - 4}$. We compute

$$\begin{aligned}
&xy\tfrac{dy}{dx} - y^2 - x\sqrt{4x^2 + y^2}\\
&= \begin{cases} x\left(x\sqrt{(\ln|x| + C)^2 - 4}\right)\left(\frac{x2(\ln|x|+C)\frac{1}{x}}{2\sqrt{(\ln|x|+C)^2-4}} + \sqrt{(\ln|x| + C)^2 - 4}\right)\\ -\left(x\sqrt{(\ln|x| + C)^2 - 4}\right)^2\\ -x\sqrt{4x^2 + \left(x\sqrt{(\ln|x| + C)^2 - 4}\right)^2}\end{cases}\\
&= \begin{cases} x^2(\ln|x| + C) + x^2((\ln|x| + C)^2 - 4)\\ -x^2(\ln|x| + C)^2 - 4)\\ -x\sqrt{4x^2 + x^2\Big((\ln|x| + C)^2 - 4\Big)}\end{cases}\\
&= \begin{cases} x^2(\ln|x| + C) + x^2(\ln|x| + C)^2 - 4x^2\\ -x^2(\ln|x| + C)^2 + 4x^2\\ -x\sqrt{4x^2 + x^2(\ln|x| + C)^2 - 4x^2}\end{cases}\\
&= \begin{cases} x^2(\ln|x| + C) + x^2(\ln|x| + C)^2\\ -x^2(\ln|x| + C)^2\\ -x^2(\ln|x| + C)\end{cases}\\
&= 0.
\end{aligned}$$

Our answer is correct.

(3) **Solve the Differential Equation**

$$x\frac{dy}{dx} + 4y - x^4y^2 = 0.$$

Please check your answer.

This is a Bernoulli Equation. Let $v = y^{1-2} = y^{-1}$. Compute $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$. Multiply both sides of the equation by $-y^{-2}$ to obtain

$$x(-y^{-2}\frac{dy}{dx}) - 4y^{-1} + x^4 = 0.$$

The Differential Equation has been transformed into

$$x\frac{dv}{dx} - 4v = -x^4.$$

Divide both sides by x:

$$\frac{dv}{dx} - \frac{4}{x}v = -x^3. \tag{1}$$

This is a First Order Linear Differential Equation. Multiply both sides of (1) by

$$\mu(x) = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = x^{-4}$$

to obtain

$$x^{-4}\frac{dv}{dx} - 4x^{-5}v = -x^{-1}. \tag{2}$$

The left side of (2) is $\frac{d}{dx}(x^{-4}v)$. Thus, equation (2) is

$$\frac{d}{dx}(x^{-4}v) = -x^{-1}. \tag{3}$$

Integrate both sides of (3) with respect to x to obtain

$$x^{-4}v = -\ln|x| + C.$$

Multiply both sides by x^4:

$$v = x^4(-\ln|x| + C).$$

Recall that $v = \frac{1}{y}$; so

$$\frac{1}{y} = x^4(-\ln|x| + C).$$

Multiply both sides by y; divide both sides by $x^4(-\ln|x| + C)$

$$\boxed{\frac{1}{x^4(C - \ln|x|)} = y}.$$

Check. We compute

$$\begin{aligned} y' &= -(x^4(C - \ln|x|))^{-2}(4Cx^3 - x^4(\tfrac{1}{x}) - 4x^3\ln|x|) \\ &= \frac{x^3(4C - 1 - 4\ln|x|)}{x^8((C - \ln|x|)^2} \end{aligned}$$

So,

$$\begin{aligned} & x\tfrac{dy}{dx} + 4y - x^4y^2 \\ &= \frac{-x^4(4C - 1 - 4\ln|x|)}{x^8(C - \ln|x|)^2} + 4\frac{1}{x^4(C - \ln|x|)} - x^4\left(\frac{1}{x^4(C - \ln|x|)}\right)^2 \\ &= \frac{(-4C + 1 + 4\ln|x|) + 4(C - \ln|x|) - 1}{x^4(C - \ln|x|)^2}. \end{aligned}$$

This is equal to zero. Our answer is correct.

(4) **Suppose an object is dropped near the surface of a planet. Gravity provides a constant acceleration of g ft/sec^2, while air resistance provides r ft/sec^2 of deceleration for each foot per second of the objects's velocity.**

(a) **Find the velocity of the object at time t. (Of course your answer will involve the positive constants g and r.)**

(b) **Find the limit as time goes to infinity of the velocity of the object.**

Let $v(t)$ be the velocity of the object at time t. We are told that

$$\frac{dv}{dt} = g - rv \quad \text{and} \quad v(0) = 0,$$

where g and r are positive constants. We separate the variables and integrate

$$\int \frac{dv}{g - rv} = \int dt$$

$$\tfrac{-1}{r}\ln|g - rv| = t + C.$$

Multiply both sides by $-r$, then exponentiate to obtain

$$\ln|g - rv| = -rt - rC,$$

$$|g - rv| = e^{-rC}e^{-rt},$$

$$g - rv = \pm e^{-rC}e^{-rt}.$$

Add rv to each side. Subtract $\pm e^{-rC}e^{-rt}$ from each side:

$$g - \pm e^{-rC}e^{-rt} = rv.$$

Divide each side by r:

$$\frac{g}{r} - \frac{\pm e^{rC}}{r} e^{-rt} = v.$$

Let K be the name for the constant $\frac{\pm e^{rC}}{r}$:

$$\frac{g}{r} - Ke^{-rt} = v.$$

Plug in $t = 0$ to learn that

$$\frac{g}{r} - K = 0.$$

Thus,

$$\boxed{\tfrac{g}{r}(1 - e^{-rt}) = v}$$

is the answer to (a) and

$$\lim_{t\to\infty} v(t) = \boxed{\tfrac{g}{r} \tfrac{\text{feet}}{\text{second}}}$$

is the answer to (b).

Check. We verify that $v = \frac{g}{r}(1 - e^{-rt})$ satisfies the Differential Equation $\frac{dv}{dt} = g - rv$.

We compute

$$\tfrac{dv}{dt} = \tfrac{d}{dv}(\tfrac{g}{r}(1 - e^{-rt})) = \tfrac{g}{r}(re^{rt}) = ge^{rt}.$$

We also compute

$$g - rv = g - r(\tfrac{g}{r}(1 - e^{-rt})) = g - g(1 - e^{-rt}) = ge^{-rt}.$$

We see that $v = \frac{g}{r}(1 - e^{-rt})$ does satisfy the Differential Equation

$$\frac{dv}{dt} = g - rv.$$

Also, $v(0) = \frac{g}{r}(1 - e^0) = 0$.

(5) **A 1500 gallon tank initially contains 600 gallons of brine, which is water with 5 lbs of salt dissolved in it. Brine, with a salt concentration of $15(1 + \cos(t))$ lbs/gal, enters the tank at a rate of 9 gal/hr. The well mixed solution leaves the tank at a rate of 6 gal/hr. Let $x(t)$ represent the number of pounds of salt in the tank at time t, where t is measured in hours. Write the Initial Value Problem whose solution is equal to $x(t)$. Please, do not solve the Initial Value Problem. Just write it down.**

$$\boxed{\frac{dx}{dt} = 9(15)(1 + \cos t) - 6\frac{x}{600 + 3t} \quad \text{and} \quad x(0) = 5.}$$

The rate of change of salt is the rate the salt enters the tank (in pounds per hour) minus the rate at which salt leaves the tank (also in pounds per hour). Salt enters the tank at 9 gallons/hour times $15(1 + \cos(t))$ pounds per gallon. Salt leaves the tank at the rate of 6 gallons/hour times the number of pounds of salt in the tank at time t divided by the number of gallons of brine in the tank at time t.