Math 242, Exam 1, Spring, 2024

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find all functions of the form $y = e^{rx}$ which are solutions of the Differential Equation $y'' + 4y' - 5y = 0$.**

If $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Plug y into the Differential Equation to obtain

$$r^2e^{rx} + 4re^{rx} - 5e^{rx} = 0.$$

So

$$e^{rx}(r^2 + 4r - 5) = 0.$$

If the product of two numbers is zero, then one of the numbers is zero. We know that e^{rx} is never zero. We conclude that $r^2 + 4r - 5 = 0$. Thus $(r - 1)(r + 5) = 0$ and $r = 1$ or $r = -5$.

$$\boxed{\text{The solutions of } y'' + 4y' - 5y = 0 \text{ of the form } y = e^{rx} \text{ are } y = e^x \text{ and } y = e^{-5x}.}$$

Check: If $y = e^x$, then $y' = e^x$ and $y'' = e^x$. When these are plugged plugged into $y'' + 4y' - 5y$, one obtains $e^x + 4e^x - 5e^x$, which is indeed zero. If $y = e^{-5x}$, then $y' = -5e^{-5x}$ and $y'' = 25e^{-5x}$. When these are plugged into $y''+4y'-5y$, one obtains $25e^{-5x}+4(-5e^{-5x})-5e^{-5x}$, which is indeed zero.

(2) (a) **Verify that $y = \frac{1}{4}x^5 + Cx^{-3}$ is a solution of the Differential Equation $x\frac{dy}{dx} + 3y = 2x^5$.**

If $y = \frac{1}{4}x^5 + Cx^{-3}$, then $y' = \frac{5}{4}x^4 - 3Cx^{-4}$. When these are plugged into $x\frac{dy}{dx} + 3y$, one obtains

$$x(\tfrac{5}{4}x^4 - 3Cx^{-4}) + 3(\tfrac{1}{4}x^5 + Cx^{-3}) = (\tfrac{5}{4} + \tfrac{3}{4})x^5 = 2x^5 \ \checkmark$$

(b) **Solve the Initial Value Problem**

$$x\frac{dy}{dx} + 3y = 2x^5 \quad \textbf{and} \quad y(2) = 1.$$

We saw in (a) that $y = \frac{1}{4}x^5 + Cx^{-3}$ is the general solution of the Differential Equation $x\frac{dy}{dx} + 3y = 2x^5$. To do (b), we must find C so that $y(2) = 1$. We must find C with $1 = \frac{1}{4}2^5 + C2^{-3}$. We want $1 = 8 + \frac{C}{8}$. We want $-7 = \frac{C}{8}$. Thus, $-56 = C$. The solution of The Initial Value problem is

$$\boxed{y = \frac{1}{4}x^5 - 56x^{-3}}.$$

(3) **Solve the Differential Equation**

$$\frac{dy}{dx} = x\sqrt{x^2+9}.$$

Separate the variables and integrate $\int dy = \int x\sqrt{x^2+9}dx$. Let $u = x^2 + 9$. Observe that $du = 2x\,dx$; so $\frac{1}{2}\,du = x\,dx$. We must integrate $\int dy = \frac{1}{2}\int u^{1/2}du$. We get $y = \frac{1}{2}\frac{2}{3}u^{3/2} + C$. The solution of the Differential Equation is

$$\boxed{y = \frac{1}{3}(x^2+9)^{3/2} + C.}$$

Check. We calculate $\frac{dy}{dx} = (\frac{1}{3})(\frac{3}{2})(2x)(x^2+9)^{1/2} = x(x^2+9)^{1/2}$. ✓

(4) **Use Euler's Method to approximate $y(1/2)$, where y is a solution of the Initial Value Problem $y' = 2y$, $y(0) = \frac{1}{2}$. Use two steps, each of size $h = \frac{1}{4}$.**

Our approximation of $y(1/2)$ is called y_2 where

$$x_0 = 0, \quad x_1 = 1/4, \quad x_2 = 1/2, \quad y_0 = \tfrac{1}{2}, \quad \text{and} \quad y_1, \quad \text{and} \quad y_2$$

are determined by

$$y_1 = y_0 + hf(x_0, y_0) \quad \text{and} \quad y_2 = y_1 + hf(x_1, y_1),$$

for $f(x,y) = 2y$. We calculate

$$y_1 = \tfrac{1}{2} + \tfrac{1}{4}2(\tfrac{1}{2}) = \tfrac{3}{4}$$

$$y_2 = \tfrac{3}{4} + \tfrac{1}{4}2(\tfrac{3}{4}) = \tfrac{3}{4} + \tfrac{3}{8} = \tfrac{9}{8}.$$

$$\boxed{\text{Our approximation of } y(1/2) \text{ is } \tfrac{9}{8}.}$$

(5) **Suppose a car starts from rest, its engine providing an acceleration of 10 ft/sec^2, while air resistance provides $.1$ ft/sec^2 of deceleration for each foot per second of the car's velocity.**

(a) **Find the velocity of the car at time t.**

(b) **Find the limit as time goes to infinity of the velocity of the car.**

Let $v(t)$ be the velocity of the car at time t. We measure t in seconds and distance in feet. Time equals zero when the car starts. We are told that $v(0) = 0$ and $\frac{dv}{dt} = 10 - \frac{1}{10}v$. We separate the variables and integrate:

$$\int \frac{1}{10 - \frac{1}{10}v} dv = \int dt.$$

Multiply top and bottom on the left by 10:

$$\int \frac{10}{100 - v} = \int dt,$$

$$-10 \ln |100 - v| = t + C.$$

Divide both sides by -10 to obtain

$$\ln |100 - v| = \frac{t}{-10} + \frac{C}{-10}.$$

Exponentiate:

$$|100 - v| = e^{\frac{C}{-10}} e^{\frac{t}{-10}},$$

$$100 - v = \pm e^{\frac{C}{-10}} e^{\frac{t}{-10}}.$$

Let K be the constant $\pm e^{\frac{C}{-10}}$.

$$100 - v = K e^{\frac{t}{-10}},$$

$$100 - K e^{\frac{t}{-10}} = v$$

Use the initial condition $v(0) = 0$ to learn that $K = 100$; so

$$\boxed{v(t) = 100(1 - e^{t/-10}) \quad \text{and} \quad \lim_{t \to \infty} v(t) = 100 \text{ ft/sec.}}$$