Problem 27 in Section 7.3. Use Laplace transforms to solve the Initial Value Problem:

$$x'' + 6x' + 25x = 0, \quad x(0) = 2, \quad x'(0) = 3.$$

Solution. Let $X = \mathcal{L}(x)$ Use $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ twice to compute

$$\mathcal{L}(x') = s\mathcal{L}(x) - x(0) = sX - 2$$

$$\mathcal{L}(x'') = s\mathcal{L}(x') - x'(0) = s(sX - 2) - 3 = s^2X - 2s - 3$$

Apply $\mathcal{L}$ to the Differential Equation and obtain

$$s^2X - 2s - 3 + 6(sX - 2) + 25X = 0$$

$$X(s^2 + 6s + 25) = 2s + 15$$

$$X = \frac{2s + 15}{s^2 + 6s + 25}$$

$$x = \mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{2s + 15}{s^2 + 6s + 25}\right)$$

The denominator does not have real number roots; so we complete the square:

$$s^2 + 6s + 25 = (s^2 + 6s + 9) + 16 = (s + 3)^2 + 16.$$

Thus

$$\begin{aligned} x &= \mathcal{L}^{-1}\left(\frac{2s + 15}{(s + 3)^2 + 16}\right) \\ &= \mathcal{L}^{-1}\left(\frac{2(s + 3) + 15 - 6}{(s + 3)^2 + 16}\right) \\ &= 2\mathcal{L}^{-1}\left(\frac{(s + 3)}{(s + 3)^2 + 16}\right) + \frac{9}{4}\mathcal{L}^{-1}\left(\frac{4}{(s + 3)^2 + 16}\right) \end{aligned}$$

We know that $\mathcal{L}^{-1}(\frac{4}{s^2+16}) = \sin 4t$; hence $\mathcal{L}^{-1}(\frac{4}{(s+3)^2+16}) = e^{-3t}\sin 4t$. In a similar manner, $\mathcal{L}^{-1}(\frac{s}{s^2+16}) = \cos 4t$; hence $\mathcal{L}^{-1}(\frac{s+3}{(s+3)^2+16}) = e^{-3t}\cos 4t$.

$$= 2e^{-3t}\cos 4t + \frac{9}{4}e^{-3t}\sin 4t$$

The solution of the initial value problem is

$$\boxed{x = 2e^{-3t}\cos 4t + \tfrac{9}{4}e^{-3t}\sin 4t}$$

Check. Plug

$$x = e^{-3t}(2\cos 4t + \tfrac{9}{4}\sin 4t)$$

$$\begin{aligned} x' &= e^{-3t}(-8\sin 4t + 9\cos 4t) - 3e^{-3t}(2\cos 4t + \tfrac{9}{4}\sin 4t) \\ &= e^{-3t}(\tfrac{-59}{4}\sin 4t + 3\cos 4t) \\ x'' &= e^{-3t}(-59\cos 4t - 12\sin 4t) - 3e^{-3t}(\tfrac{-59}{4}\sin 4t + 3\cos 4t) \\ &= e^{-3t}(-68\cos 4t + \tfrac{129}{4}\sin 4t) \end{aligned}$$

into $x'' + 6x' + 25x$ and obtain

$$\begin{cases} e^{-3t}(-68\cos 4t + \tfrac{129}{4}\sin 4t) \\ +6e^{-3t}(\tfrac{-59}{4}\sin 4t + 3\cos 4t) \\ +25e^{-3t}(2\cos 4t + \tfrac{9}{4}\sin 4t) \end{cases}$$

$$= (-68 + 18 + 50)\cos 4t + \tfrac{1}{4}(129 - 6(59) + 9(25))\sin 4t = 0; \checkmark$$

$x(0) = 2\checkmark$; $x'(0) = 4\checkmark$. Our proposed answer does everything it is supposed to do; it is correct.