Problem 7 in Section 7.2. Use Laplace transforms to solve the Initial Value Problem

$$x'' + x = \cos 3t, \quad x(0) = 1, \quad x'(0) = 0.$$

Solution. Let $X = \mathcal{L}(x)$. It follows that

$$\begin{aligned}\mathcal{L}(x') &= s\mathcal{L}(x) - x(0) = sX - 1 \\ \mathcal{L}(x'') &= s\mathcal{L}(x') - x'(0) = s(sX-1) - 0 = s^2X - s\end{aligned}$$

Apply $\mathcal{L}$ to $x'' + x = \cos 3t$ and obtain

$$\begin{aligned}\mathcal{L}(x'') + \mathcal{L}(x) &= \mathcal{L}(\cos 3t) \\ s^2X - s + X &= \frac{s}{s^2+9} \\ (s^2+1)X &= s + \frac{s}{s^2+9} \\ X &= \frac{s}{s^2+1} + \frac{s}{(s^2+9)(s^2+1)} \qquad (22)\end{aligned}$$

We apply the technique of partial fractions to write $\frac{s}{(s^2+9)(s^2+1)}$ in the form

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}.$$

(Keep in mind that we know how to find the inverse Laplace transform of the most recent expression.) We look for constants A, B, C, and D so that

$$\frac{s}{(s^2+9)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}.$$

m Multiply both sides by $(s^2+9)(s^2+1)$.

$$s = (As+B)(s^2+9) + (Cs+D)(s^2+1)$$

$$s = (A+C)s^3 + (B+D)s^2 + (9A+C)s + (9B+D)$$

We want

$$\begin{cases} 0 = A + C \\ 0 = B + D \\ 1 = 9A + C \\ 0 = 9B + D \end{cases}$$

Equation two tells us that $B = -D$ and equation 4 tells us that $0 = -9D + D$; hence, $D = B = 0$. Equation 1 tells us that $C = -A$ and equation 3 tells us that $1 = 8A$. Thus $A = \frac{1}{8}$ and $C = -\frac{1}{8}$. Equation (22) now is

$$X = \frac{s}{s^2+1} + \frac{1}{8}\left[\frac{s}{s^2+1} - \frac{s}{s^2+9}\right]$$

$$X = \frac{1}{8}\left[9\frac{s}{s^2+1} - \frac{s}{s^2+9}\right]$$

$$x = \mathcal{L}^{-1}(X) = \frac{9}{8}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{8}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right)$$

$$\boxed{x = \tfrac{9}{8}\cos t - \tfrac{1}{8}\cos 3t.}$$

Check. Plug

$$\begin{aligned} x &= \tfrac{9}{8}\cos t - \tfrac{1}{8}\cos 3t \\ x' &= -\tfrac{9}{8}\sin t + \tfrac{3}{8}\sin 3t \\ x'' &= -\tfrac{9}{8}\cos t + \tfrac{9}{8}\cos 3t \end{aligned}$$

into $x'' + x$ and obtain

$$\begin{cases} -\tfrac{9}{8}\cos t + \tfrac{9}{8}\cos 3t \\ +\tfrac{9}{8}\cos t - \tfrac{1}{8}\cos 3t \end{cases}$$

$= \cos 3t\checkmark$; $x(0) = \frac{9}{8} - \frac{1}{8} = 1\checkmark$; and $x'(0) = 0 + 0 = 0\checkmark$. Our proposed answer does everything it is supposed to do. It is correct.