Problem 31 in Section 7.2. In class we calculated

$$\mathcal{L}(t\cos kt) = \frac{s^2-k^2}{(s^2+k^2)^2} \quad \text{and} \quad \mathcal{L}(\sin kt) = \frac{k}{(s^2+k^2)}.$$

Use these facts to calculate

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+k^2)^2}\right).$$

Solution. We get a common denominator:

$$\mathcal{L}(\sin kt) = \frac{k(s^2+k^2)}{(s^2+k^2)^2} \quad \text{and} \quad \mathcal{L}(kt\cos kt) = \frac{k(s^2-k^2)}{(s^2+k^2)^2}.$$

If we calculate the first one minus the second one we get

$$\mathcal{L}(\sin kt - kt\cos kt) = \frac{k(s^2+k^2)}{(s^2+k^2)^2} - \frac{k(s^2-k^2)}{(s^2+k^2)^2} = \frac{2k^3}{(s^2+k^2)^2}.$$

Divide both sides by the constant $2k^3$ to obtain

$$\frac{1}{2k^3}\mathcal{L}(\sin kt - kt\cos kt) = \frac{1}{(s^2+k^2)^2}.$$

Of course $k\mathcal{L}(f) = \mathcal{L}(kf)$ whenever k is a constant and f is a function; so

$$\mathcal{L}\left(\frac{1}{2k^3}(\sin kt - kt\cos kt)\right) = \frac{1}{(s^2+k^2)^2}$$

and

$$\boxed{\frac{1}{2k^3}(\sin kt - kt\cos kt) = \mathcal{L}^{-1}\left(\frac{1}{(s^2+k^2)^2}\right).}$$