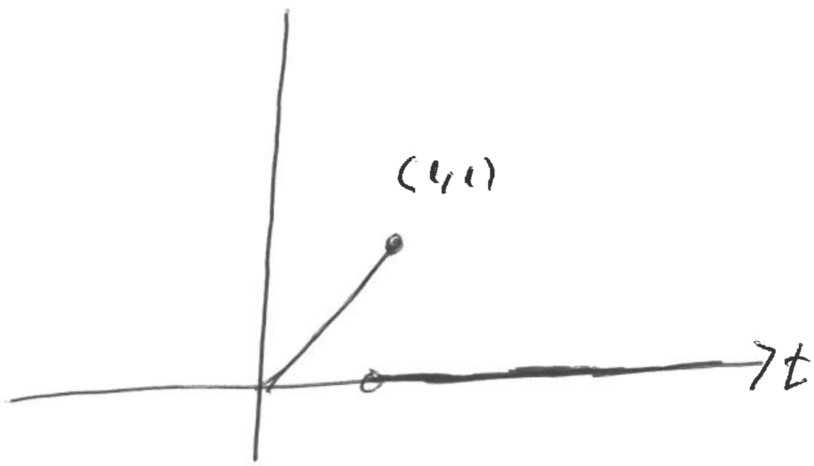
Problem 9 in Section 7.1. Compute the Laplace transform of the function $f(t)$ whose picture is on the next page.

Section 7.1 Problem 9 Picture



Solution. We see that

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } 1 < t. \end{cases}$$

Recall that $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$. We compute

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)\, dt \\
&= \int_0^1 e^{-st} t\, dt + \int_1^\infty e^{-st} 0\, dt
\end{aligned}$$

We use integration by parts:

$$\int u\, dv = uv - \int v\, du.$$

Take $u = t$ and $dv = e^{-st} dt$; compute $du = dt$ and $v = \frac{1}{-s} e^{-st}$.

$$\begin{aligned}
&= \left[uv - \int v du \right]_{t=0}^{t=1} \\
&= \left[\frac{t}{-s} e^{-st} - \int \frac{1}{-s} e^{-st} dt \right]_{t=0}^{t=1} \\
&= \left[\frac{t}{-s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_{t=0}^{t=1} \\
&= \left[\frac{1}{-s} e^{-s1} - \frac{1}{s^2} e^{-s1} \right] - \left[\frac{(0)}{-s} e^{-s(0)} - \frac{1}{s^2} e^{-s(0)} \right] \\
&= \boxed{\frac{1}{-s} e^{-s} - \frac{1}{s^2} e^{-s} + \frac{1}{s^2}}
\end{aligned}$$