Problem 1 in Section 7.1. Use the definition of $\mathcal{L}$ to compute $\mathcal{L}(f(t))$ for $f(t) = t$.

Solution. Recall that $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$. We must compute

$$\mathcal{L}(t) = \int_0^\infty e^{-st} t\, dt.$$

We use integration by parts:

$$\int u\, dv = uv - \int v\, du.$$

Take $u = t$ and $dv = e^{-st} dt$. Compute $du = dt$ and $v = \frac{1}{-s} e^{-st}$. It follows that

$$\begin{aligned}
\mathcal{L}(t) &= \int_0^\infty e^{-st} t\, dt \\
&= \Big[(uv - \int v du\Big]\Big|_{t=0}^{t=\infty} \\
&= \Big[-\frac{t}{s}e^{-st} - \int -\frac{1}{s}e^{-st} dt\Big]_{t=0}^{\infty} \\
&= \Big[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big]_{t=0}^{\infty}
\end{aligned}$$

Of course, integrate to infinity means integrate to a number b and take the limit as b goes to infinity.

$$= \lim_{b\to\infty} \Big[-\frac{b}{s}e^{-sb} - \frac{1}{s^2}e^{-sb}\Big] - \Big[-\frac{0}{s}e^{-s(0)} - \frac{1}{s^2}e^{-s(0)}\Big]$$

It is obvious that $\lim_{b\to\infty} \frac{1}{e^{sb}} = 0$, when s is a positive constant. Use L'Hopital's rule, if necessary to see that $\lim_{b\to\infty} \frac{b}{e^{sb}} = 0$, when s is a positive constant.

$$= 0 + 0 + 0 + \frac{1}{s^2} = \boxed{\frac{1}{s^2} \text{ provided } 0 < s}$$