Problem 39 in Section 3.5. Solve the Initial Value Problem

$$y''' + y'' = x + e^{-x}, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1.$$

Solution.

There are three steps.

First we solve the homogeneous problem. That is, we find the general solution of $y''' + y'' = 0$. We try $y = e^{rx}$ and we consider the characteristic equation

$$r^3 + r^2 = 0$$
$$r^2(r+1) = 0$$

The general solution of the homogeneous problem is

$$y = c_1 + c_2 x + c_3 e^{-x}. \tag{17}$$

Then we find a particular solution of the given Differential Equation.

Actually we will do this steps in two pieces. First we will find a particular solution for

$$y''' + y'' = x,$$

then we will find a particular solution for $y''' + y'' = e^{-x}$, then we will add these two particular solutions to get a particular solution for $y''' + y'' = x + e^{-x}$.

We find a particular solution for $y''' + y'' = x$. Normally, if we want to get a linear expression out of a linear differential operator with constant coefficients, we would input a linear expression like $Ax + B$. But the linear operator

$$\frac{d^3}{dx^3} + \frac{d^2}{dx^2}$$

sends all linear expressions $Ax+B$; so we better raise the powers on x. Let us try for a particular solution of $y''' + y'' = x$ of the form $y = Ax^3 + Bx^2$. (I think we need to play with two consecutive powers of x. It is clear that $Ax + B$ won't work. It is very unlikely that $Ax^2 + Bx$ will work; because $y''' + y''$ sends Bx to zero automatically. So $Ax^2 + Bx$ really isn't two consecutive interesting powers of x. At any rate, if you try the wrong candidate, there won't be any solution, then you merely try a better candidate.)

Plug

$$\begin{aligned} y &= Ax^3 + Bx^2 \\ y' &= 3Ax^2 + 2Bx \\ y'' &= 6Ax + 2B \\ y''' &= 6A \end{aligned}$$

into

$$y''' + y'' = x$$

and obtain

$$6A + (6Ax + 2B) = x.$$

We want to choose A and B so that

$$(6A + 2B) + 6Ax = x.$$

We want $0 = 6A + 2B$ and $6A = 1$. We take $A = \frac{1}{6}$ and $B = -\frac{1}{2}$. That is,

$$y = \frac{1}{6}x^3 - \frac{1}{2}x^2 \text{ is a particular solution of } y''' + y'' = x \tag{18}$$

We find a particular solution for $y''' + y'' = e^{-x}$. Normally, we would look for a solution of the form $y = Ae^{-x}$; but we know that all functions of the form $y = Ae^{-x}$ are solutions of the form $y''' + y'' = 0$; so we won't any functions of the form $y = Ae^{-x}$ which are solutions of $y''' + y'' = e^{-x}$. So, instead we look for a solution of $y''' + y'' = e^{-x}$ of the form $y = Axe^{-x}$. Plug

$$\begin{aligned}
y &= Axe^{-x} \\
y' &= -Axe^{-x} + Ae^{-x} = A(1-x)e^{-x} \\
y'' &= -A(1-x)e^{-x} - Ae^{-x} = A(-2+x)e^{-x} \\
y''' &= -A(-2+x)e^{-x} + Ae^{-x} = A(3-x)e^{-x}
\end{aligned}$$

into $y''' + y'' = e^{-x}$ and obtain

$$A(3-x)e^{-x} + A(-2+x)e^{-x} = e^{-x}$$

$$Ae^{-x} = e^{-x}$$

We take $A = 1$. Thus,

$$y = xe^{-x} \text{ is a solution of } y''' + y'' = e^{-x}. \tag{19}$$

Combine (17), (18), and (19) to see that

$$y = c_1 + c_2x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + c_3e^{-x} + xe^{-x}$$

is the general solution of $y''' + y'' = x + e^{-x}$.

Step 3. We use the initial condition to evaluate the constants. Plug $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 1$ into

$$\begin{aligned}
y &= c_1 + c_2x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + c_3e^{-x} + xe^{-x} \\
y' &= c_2 - x + \tfrac{1}{2}x^2 - c_3e^{-x} - xe^{-x} + e^{-x} \\
&= c_2 - x + \tfrac{1}{2}x^2 + (1-c_3)e^{-x} - xe^{-x} \\
y'' &= -1 + x - (1-c_3)e^{-x} + xe^{-x} - e^{-x}
\end{aligned}$$

and obtain

$$\begin{cases} 1 = c_1 + c_3 \\ 0 = c_2 - c_3 + 1 \\ 1 = -1 - (1 - c_3) - 1 \end{cases}$$

The bottom equation gives $c_3 = 4$. The middle equation gives $c_2 = 3$. The top equation gives that $c_1 = -3$. We conclude that

$$\boxed{y = -3 + 3x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + 4e^{-x} + xe^{-x}}$$

is the solution of the Initial Value Problem

$$y''' + y'' = x + e^{-x}, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1.$$

Check. Plug

$$\begin{aligned} y &= -3 + 3x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + 4e^{-x} + xe^{-x} \\ y' &= 3 - x + \tfrac{1}{2}x^2 - 4e^{-x} - xe^{-x} + e^{-x} \\ &= 3 - x + \tfrac{1}{2}x^2 - 3e^{-x} - xe^{-x} \\ y'' &= -1 + x + 3e^{-x} + xe^{-x} - e^{-x} \\ &= -1 + x + 2e^{-x} + xe^{-x} \\ y''' &= 1 - 2e^{-x} - xe^{-x} + e^{-x} \\ &= 1 - e^{-x} - xe^{-x} \end{aligned}$$

into $y''' + y''$ and obtain

$$\Big(1 - e^{-x} - xe^{-x}\Big) + \Big(-1 + x + 2e^{-x} + xe^{-x}\Big) = x + e^{x}\checkmark;$$

$y(0) = -3 + 4 = 1\checkmark$; $y'(0) = 3 - 3 = 0\checkmark$; and $y''(0) = -1 + 3 - 1 = 1\checkmark$. Our proposed solution does everything it is supposed to do. It is correct.