Problem 33 in Section 3.5. Solve the initial value problem

$$y'' + 9y = \sin 2x, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution. There are three steps.

First we solve the homogeneous problem $y'' + 9y = 0$.

We try $y = e^{rx}$ and we consider the Characteristic Equation

$$r^2 + 9 = 0.$$

We see that $r = 3i$ or $r = -3i$. The general solution of the homogeneous equation is $y = C_1 \sin 3x + C_2 \cos 3x$.

Then we find a particular solution of the given Differential Equation.

We try $y = A \sin 2x + B \cos 2x$. We plug

$$\begin{aligned} y &= A \sin 2x + B \cos 2x \\ y' &= 2A \cos 2x - 2B \sin 2x \\ y'' &= -4A \sin 2x - 4B \cos 2x \end{aligned}$$

into $y'' + 9y = \sin 2x$ to obtain

$$-4A \sin 2x - 4B \cos 2x + 9\Big(A \sin 2x + B \cos 2x\Big) = \sin 2x$$

$$(-4A + 9A) \sin 2x + (-4B + 9B) \cos 2x = \sin 2x$$

$$(5A) \sin 2x + (5B) \cos 2x = \sin 2x$$

We take $5A = 1$ and $5B = 0$. That is, $A = \frac{1}{5}$ and $B = 0$. The general solution of the Differential Equation is

$$y = C_1 \sin 3x + C_2 \cos 3x + \tfrac{1}{5} \sin 2x.$$

Step 3. We use the initial condition to evaluate the constants.

We compute $y' = 3C_1 \cos 3x - 3c_2 \sin 3x + \frac{2}{5} \cos 2x$. Use $y(0) = 1$ and $y'(0) = 0$ to see that

$$1 = C_2 \quad \text{and} \quad 0 = 3C_1 + \frac{2}{5}.$$

Thus, $C_1 = -\frac{2}{15}$ and $C_2 = 1$. The solution of the Initial Value Problem is

$$\boxed{y = -\tfrac{2}{15} \sin 3x + \cos 3x + \tfrac{1}{5} \sin 2x.}$$

Check. We plug

$$\begin{aligned} y &= -\tfrac{2}{15}\sin 3x + \cos 3x + \tfrac{1}{5}\sin 2x \\ y' &= -\tfrac{6}{15}\cos 3x - 3\sin 3x + \tfrac{2}{5}\cos 2x \\ y'' &= +\tfrac{18}{15}\sin 3x - 9\cos 3x - \tfrac{4}{5}\sin 2x \end{aligned}$$

into $y'' + 9y$ and obtain

$$\Big(+ \tfrac{18}{15}\sin 3x - 9\cos 3x - \tfrac{4}{5}\sin 2x\Big) + 9\Big(- \tfrac{2}{15}\sin 3x + \cos 3x + \tfrac{1}{5}\sin 2x\Big)$$

$$= (\tfrac{6}{5} - \tfrac{6}{5})\sin 3x + (-9 + 9)\cos 3x + (-\tfrac{4}{5} + \tfrac{9}{5})\sin 2x = \sin 2x, \checkmark$$

$y(0) = 1\checkmark$ and $y'(0) = -\frac{6}{15} + \frac{2}{5} = 0.\checkmark$