Problem 17 in Section 3.4. Solve the Initial Problem

$$x'' + 8x' + 16x = 0, \quad x(0) = 5, \quad x'(0) = -10.$$

Put your answer in the form $x(t) = Ce^{at}\cos(bt - \alpha)$ if this makes sense. Sketch the graph of $x = x(t)$.

Solution. We try $x = e^{rt}$. We must study the characteristic equation

$$r^2 + 8r + 16 = 0$$

$$(r+4)^2 = 0.$$

We have a repeated root of $r = -4$. The general solution of the Differential equation is

$$x = c_1 e^{-4t} + c_2 t e^{-4t}.$$

We re-write this solution as

$$x = (c_1 + c_2 t)e^{-4t}.$$

We take the derivative in order to evaluate the constants.

$$x' = -4(c_1 + c_2 t)e^{-4t} + c_2 e^{-4t}$$

$$x' = (-4c_1 + c_2 - 4c_2 t)e_{-4t}$$

Plug $t = 0$ into $x(t)$ and $x'(t)$ to see that

$$\begin{aligned} 5 &= \quad c_1 \\ -10 &= -4c_1 + c_2 \end{aligned}$$

Thus, $c_1 = 5$ and $c_2 = 10$. The solution is

$$\boxed{x = (5 + 10t)e^{-4t}.}$$

Check. Plug

$$\begin{aligned} x &= (5 + 10t)e^{-4t} \\ x' &= -4(5 + 10t)e^{-4t} + 10e^{-4t} \\ &= (-10 - 40t)e^{-4t} \\ x'' &= -4(-10 - 40t)e^{-4t} - 40e^{-4t} \\ &= 160te^{-4t} \end{aligned}$$

into $x'' + 8x' + 16x$ and obtain

$$160te^{-4t} + 8((-10 - 40t)e^{-4t}) + 16((5 + 10t)e^{-4t})$$

$$= \Big((-80 + 80) + (160 - 320 + 160)t\Big)e^{-4t} = 0, \checkmark$$

$x(0) = 5\checkmark$, $x'(0) = -10\checkmark$.

Our solution does everything it is supposed to do. It is correct.

Picture for Section 3.4 Problem 17

We are supposed to graph $x = (5 + 10t)e^{-4t}$.

We know that $x' = (-10 - 40t)e^{-4t}$ is negative for $0 < t$

We know that $x'' = 160t\,e^{-4t}$ is positive for $0 < t$

We know $\lim_{t \to \infty} x = 0$ (use L'hopital's rule if necessary)

So the graph is always decreasing, always concave up, the graph starts at $t=0$ $x=5$ and the graph approaches the t-axis as t goes to infinity.