Problem 15 in Section 3.4. Solve the Initial Problem

$$\tfrac{1}{2}x'' + 3x' + 4x = 0, \quad x(0) = 2, \quad x'(0) = 0.$$

Put your answer in the form $x(t) = Ce^{at}\cos(bt - \alpha)$ if this makes sense. Sketch the graph of $x = x(t)$.

Solution. We try $x = e^{rt}$. We must study the characteristic polynomial

$$\frac{1}{2}r^2 + 3r + 4 = 0.$$

Multiply both sides by 2 and factor:

$$r^2 + 6r + 8 = 0$$

$$(r+2)(r+4) = 0$$

So $r = -2$ and $r = -4$ are the roots of the characteristic polynomial. The general solution of the Differential Equation is

$$x = c_1 e^{-2x} + c_2 e^{-4x}.$$

We use the initial conditions to evaluate the constants. We compute

$$\begin{aligned} x &= c_1 e^{-2x} + c_2 e^{-4x} \\ x' &= -2c_1 e^{-2x} - 4c_2 e^{-4x} \end{aligned}$$

It follows that

$$\begin{aligned} 2 &= c_1 + c_2 \\ 0 &= -2c_1 - 4c_2 \end{aligned}$$

Replace Equation 2 with Equation 2 plus 2 times Equation 1:

$$\begin{aligned} 2 &= c_1 + c_2 \\ 4 &= \qquad -2c_2 \end{aligned}$$

We conclude that $c_2 = -2$ and $c_1 = 4$. Thus,

$$\boxed{x = 4e^{-2t} - 2e^{-4t}}$$

When we graph this curve, we might want to factor out the bigger term:

$$x = 4e^{-2t}(1 - \frac{1}{2}e^{-2t})$$

We notice the the factor $(1 - \frac{1}{2}e^{-2t})$ starts at $\frac{1}{2}$ (when $t = 0$) and degrees towards 0. So the graph of

$$x = 4e^{-2t}(1 - \frac{1}{2}e^{-2t})$$

looks a lot like, and lives under, the graph of $x = 4e^{-2t}$. We know the graph of the exponential function $x = 4e^{-2t}$.

Check. We plug

$$\begin{aligned} x &= 4e^{-2t} - 2e^{-4t} \\ x' &= -8e^{-2t} + 8e^{-4t} \\ x'' &= 16e^{-2t} - 32e^{-4t} \end{aligned}$$

into $\frac{1}{2}x'' + 3x' + 4x$ and obtain

$$\begin{aligned} &(1/2)(16e^{-2t} - 32e^{-4t}) \\ &+ 3(-8e^{-2t} + 8e^{-4t}) \\ &+ 4(4e^{-2t} - 2e^{-4t}) \end{aligned}$$

$$= e^{-2t}(8 - 24 + 16) + e^{-4t}(-16 + 24 - 8) = 0\checkmark.$$

We also see that $x(0) = 4 - 2 = 2\checkmark$ and $x'(0) = -8 + 8 = 0$.

Our proposed solution does everything it is supposed to do. It is correct.

The picture is on the next page.

Picture for 3.4 #15

We are supposed to draw $X = 4e^{-2t} - 2e^{-4t}$

$$X = 4e^{-2t}\left(1 - \tfrac{1}{2}e^{-2t}\right)$$

Notice that $1 - \frac{1}{2}e^{-2t}$ is equal to $\frac{1}{2}$ when $t = 0$

and as t grows toward infinity $1 - \frac{1}{2}e^{-2t}$ grows to 1

The graph we are supposed to draw is always a positive fraction of $X = 4e^{-2t}$