Problem 35 in Section 3.3. Find the general solution of

$$6y'''' + 5y''' + 25y'' + 20y' + 4y = 0.$$

Hint: one solution is $y = \cos(2x)$.

Solution. We try $y = e^{rx}$. We plug y, $y' = re^{rx}$, $y'' = r^2e^{rx}$, $y''' = r^3e^{rx}$, and $y'''' = r^4e^{rx}$ into the Differential Equation. We want

$$6r^4e^{rx} + 5r^3e^{rx} + 25r^2e^{rx} + 20re^{rx} + 4e^{rx} = 0.$$

We want

$$e^{rx}(6r^4 + 5r^3 + 25r^2 + 20r + 4) = 0.$$

If a product is zero, one of the factors must be zero. The function e^{rx} is never zero; so we want

$$6r^4 + 5r^3 + 25r^2 + 20r + 4 = 0.$$

The hint that $y = \cos(2x)$ is a solution of the Differential Equation tells us that $r - 2i$ is a factor of $r^4 + 5r^3 + 25r^2 + 20r + 4$. Of course, $r + 2i$ is also a factor; so $(r-2i)(r+2i) = r^2+4$ is a factor of $6r^4+5r^3+25r^2+20r+4$. Use long division (or do it in your head) to see that the other factor is $6r^2+5r+1$. So

$$6r^4+5r^3+25r^2+20r+4 = (6r^2+5r+1)(r^2+1) = (2r+1)(3r+1)(r-2i)(r+2i)$$

and r is equal to $-\frac{1}{2}$, $-\frac{1}{3}$, $2i$, $-2i$. The corresponding solutions of the Differential Equation are $y = e^{-\frac{1}{2}x}$, $y = e^{-\frac{1}{3}x}$, $y = \cos 2x$, and $y = \sin 2x$. The general solution of the Differential Equation is

$$\boxed{y = c_1e^{-\frac{1}{2}x} + c_2e^{-\frac{1}{3}x} + c_3\cos 2x + c_4\sin 2x}.$$

Check. Plug

$$\begin{aligned}
y &= c_1e^{-\frac{1}{2}x} + c_2e^{-\frac{1}{3}x} + c_3\cos 2x + c_4\sin 2x\\
y' &= (-\tfrac{1}{2})c_1e^{-\frac{1}{2}x} + (-\tfrac{1}{3})c_2e^{-\frac{1}{3}x} - 2c_3\sin 2x + 2c_4\cos 2x\\
y'' &= (\tfrac{1}{4})c_1e^{-\frac{1}{2}x} + (\tfrac{1}{9})c_2e^{-\frac{1}{3}x} - 4c_3\cos 2x - 4c_4\sin 2x\\
y''' &= (-\tfrac{1}{8})c_1e^{-\frac{1}{2}x} + (-\tfrac{1}{27})c_2e^{-\frac{1}{3}x} + 8c_3\sin 2x - 8c_4\cos 2x\\
y'''' &= (\tfrac{1}{16})c_1e^{-\frac{1}{2}x} + (\tfrac{1}{81})c_2e^{-\frac{1}{3}x} + 16c_3\cos 2x + 16c_4\sin 2x
\end{aligned}$$

into $6y'''' + 5y''' + 25y'' + 20y' + 4y$ and obtain

$$\begin{cases}
+6\Big((\tfrac{1}{16})c_1e^{-\frac{1}{2}x} + (\tfrac{1}{81})c_2e^{-\frac{1}{3}x} + 16c_3\cos 2x + 16c_4\sin 2x\Big)\\
+5\Big((-\tfrac{1}{8})c_1e^{-\frac{1}{2}x} + (-\tfrac{1}{27})c_2e^{-\frac{1}{3}x} + 8c_3\sin 2x - 8c_4\cos 2x\Big)\\
+25\Big((\tfrac{1}{4})c_1e^{-\frac{1}{2}x} + (\tfrac{1}{9})c_2e^{-\frac{1}{3}x} - 4c_3\cos 2x - 4c_4\sin 2x\Big)\\
+20\Big(-\tfrac{1}{2})c_1e^{-\frac{1}{2}x} + (-\tfrac{1}{3})c_2e^{-\frac{1}{3}x} - 2c_3\sin 2x + 2c_4\cos 2x\Big)\\
+4\Big(c_1e^{-\frac{1}{2}x} + c_2e^{-\frac{1}{3}x} + c_3\cos 2x + c_4\sin 2x\Big)
\end{cases}$$

$$= \begin{cases} +\left(\frac{6}{16} - \frac{5}{8} + \frac{25}{4} - 10 + 4\right)c_1 e^{-\frac{1}{2}x} \\ +\left(\frac{6}{81} - \frac{5}{27} + \frac{25}{9} - \frac{20}{3} + 4\right)c_2 e^{-\frac{1}{3}x} \\ +\Big(6(16) + 5(8) - 25(4) - 20(2) + 4\Big)c_3 \cos 2x \\ +\Big(6(16) - 5(8) - 25(4) + 20(2) + 4\Big)c_3 \sin 2x \end{cases}$$

$$= 0\checkmark.$$