Problem 17 in Section 3.3. Find the general solution of $6y''''+11y''+4y=0$.

Solution. We try $y=e^{rx}$. We plug y, $y'=re^{rx}$, $y''=r^2e^{rx}$, $y'''=r^3e^{rx}$, and $y''''=r^4e^{rx}$ into the Differential Equation. We want

$$6r^4e^{rx}+11r^2e^{rx}+4e^{rx}=0.$$

We want $e^{rx}(6r^4+11r^2+4)=0$. If a product is zero, one of the factors must be zero. The function e^{rx} is never zero; so we want

$$(6r^4+11r^2+4)=0$$
$$(2r^2+1)(3r^2+4)=0$$
$$r^2=-\tfrac{1}{2},\text{ or } r^2=-\tfrac{4}{3}$$
$$r=\pm\tfrac{1}{\sqrt{2}}i \quad\text{or}\quad \pm\tfrac{2}{\sqrt{3}}i$$

Recall that if $r=a+bi$ is a root of the characteristic polynomial, then $y=e^{ax}\cos(bx)$ and $y=e^{ax}\sin(bx)$ both are solutions of the corresponding linear homogeneous Differential Equation with constant coefficients. Thus,

$$y=\cos(\tfrac{1}{\sqrt{2}}x),\quad y=\sin(\tfrac{1}{\sqrt{2}}x),\quad \cos(\tfrac{2}{\sqrt{3}}x),\quad\text{and}\quad y=\sin(\tfrac{2}{\sqrt{3}}x)$$

are four linearly independent solutions of $6y''''+11y''+4y=0$. The general solution of this Differential Equation is

$$\boxed{y=c_1\cos(\tfrac{1}{\sqrt{2}}x)+c_2\sin(\tfrac{1}{\sqrt{2}}x)+c_3\cos(\tfrac{2}{\sqrt{3}}x)+c_4\sin(\tfrac{2}{\sqrt{3}}x).}$$

Check. We plug

$$\begin{aligned}
y&=c_1\cos(\tfrac{1}{\sqrt{2}}x)+c_2\sin(\tfrac{1}{\sqrt{2}}x)+c_3\cos(\tfrac{2}{\sqrt{3}}x)+c_4\sin(\tfrac{2}{\sqrt{3}}x)\\
y'&=-\tfrac{1}{\sqrt{2}}c_1\sin(\tfrac{1}{\sqrt{2}}x)+\tfrac{1}{\sqrt{2}}c_2\cos(\tfrac{1}{\sqrt{2}}x)-\tfrac{2}{\sqrt{3}}c_3\sin(\tfrac{2}{\sqrt{3}}x)+\tfrac{2}{\sqrt{3}}c_4\cos(\tfrac{2}{\sqrt{3}}x)\\
y''&=-(\tfrac{1}{\sqrt{2}})^2c_1\cos(\tfrac{1}{\sqrt{2}}x)-(\tfrac{1}{\sqrt{2}})^2c_2\sin(\tfrac{1}{\sqrt{2}}x)-(\tfrac{2}{\sqrt{3}})^2c_3\cos(\tfrac{2}{\sqrt{3}}x)-(\tfrac{2}{\sqrt{3}})^2c_4\sin(\tfrac{2}{\sqrt{3}}x)\\
y'''&=+(\tfrac{1}{\sqrt{2}})^3c_1\sin(\tfrac{1}{\sqrt{2}}x)-(\tfrac{1}{\sqrt{2}})^3c_2\cos(\tfrac{1}{\sqrt{2}}x)+(\tfrac{2}{\sqrt{3}})^3c_3\sin(\tfrac{2}{\sqrt{3}}x)-(\tfrac{2}{\sqrt{3}})^3c_4\sin(\tfrac{2}{\sqrt{3}}x)\\
y''''&=+(\tfrac{1}{\sqrt{2}})^4c_1\cos(\tfrac{1}{\sqrt{2}}x)+(\tfrac{1}{\sqrt{2}})^4c_2\sin(\tfrac{1}{\sqrt{2}}x)+(\tfrac{2}{\sqrt{3}})^4c_3\cos(\tfrac{2}{\sqrt{3}}x)+(\tfrac{2}{\sqrt{3}})^4c_4\sin(\tfrac{2}{\sqrt{3}}x)
\end{aligned}$$

into $6y''''+11y''+4y$ and obtain

$$\begin{cases}
+6\Big(+(\tfrac{1}{\sqrt{2}})^4c_1\cos(\tfrac{1}{\sqrt{2}}x)+(\tfrac{1}{\sqrt{2}})^4c_2\sin(\tfrac{1}{\sqrt{2}}x)+(\tfrac{2}{\sqrt{3}})^4c_3\cos(\tfrac{2}{\sqrt{3}}x)+(\tfrac{2}{\sqrt{3}})^4c_4\sin(\tfrac{2}{\sqrt{3}}x)\Big)\\
+11\Big(-(\tfrac{1}{\sqrt{2}})^2c_1\cos(\tfrac{1}{\sqrt{2}}x)-(\tfrac{1}{\sqrt{2}})^2c_2\sin(\tfrac{1}{\sqrt{2}}x)-(\tfrac{2}{\sqrt{3}})^2c_3\cos(\tfrac{2}{\sqrt{3}}x)-(\tfrac{2}{\sqrt{3}})^2c_4\sin(\tfrac{2}{\sqrt{3}}x)\Big)\\
+4\Big(c_1\cos(\tfrac{1}{\sqrt{2}}x)+c_2\sin(\tfrac{1}{\sqrt{2}}x)+c_3\cos(\tfrac{2}{\sqrt{3}}x)+c_4\sin(\tfrac{2}{\sqrt{3}}x)\Big)
\end{cases}$$

$$=(\tfrac{6}{4}-\tfrac{11}{2}+4)\Big(c_1\cos(\tfrac{1}{\sqrt{2}}x)+c_2\sin(\tfrac{1}{\sqrt{2}}x)\Big)+(\tfrac{16(6)}{9}-\tfrac{44}{3}+4)\Big(c_3\cos(\tfrac{2}{\sqrt{3}}x)+c_4\sin(\tfrac{2}{\sqrt{3}}x)\Big)$$
$$=0.$$

✓Our answer is correct.