Problem 11 in Section 3.3. Find the general solution of $y^{(4)} - 8y^{(3)} + 16y'' = 0$.

Solution. We try $y = e^{rx}$. We plug y, $y' = re^{rx}$, $y'' = r^2e^{rx}$, $y''' = r^3e^{rx}$, and $y'''' = r^4e^{rx}$ into the Differential Equation. We want

$$r^4e^{rx} - 8r^3e^{rx} + 16r^2e^{rx} = 0.$$

We want $e^{rx}(r^4 - 8r^3 + 16r^2) = 0$. If a product is zero, one of the factors must be zero. The function e^{rx} is never zero; so we want

$$(r^4 - 8r^3 + 16r^2) = 0$$

$$r^2(r^2 - 8r + 16) = 0$$
$$r^2(r-4)^2 = 0$$

So r is 0 (with multiplicity 2) or r is 4 (also with multiplicity 2)., Four linearly independent solutions of the Differential Equation are 1, x, e^{4x}, and xe^{4x}. (Of course $1 = e^{0x}$ and $x = xe^{0x}$.) The general solution of the Differential Equation is

$$\boxed{y = c_1 + c_2x + c_3e^{4x} + c_4xe^{4x}.}$$

Check. We plug

$$\begin{aligned}
y &= c_1 + c_2x + c_3e^{4x} + c_4xe^{4x}\\
y' &= \quad c_2 + 4c_3e^{4x} + c_4e^{4x} + 4c_4xe^{4x}\\
&= \quad c_2 + (4c_3 + c_4)e^{4x} + 4c_4xe^{4x}\\
y'' &= \qquad 4(4c_3 + c_4)e^{4x} + 4c_4e^{4x} + 16c_4xe^{4x}\\
&= \qquad (16c_3 + 8c_4)e^{4x} + 16c_4xe^{4x}\\
y''' &= \qquad 4(16c_3 + 8c_4)e^{4x} + 16c_4e^{4x} + 64c_4xe^{4x}\\
&= \qquad (64c_3 + 48c_4)e^{4x} + 64c_4xe^{4x}\\
y'''' &= \qquad 4(64c_3 + 48c_4)e^{4x} + 64c_4e^{4x} + 256c_4xe^{4x}\\
&= \qquad (256c_3 + 256c_4)e^{4x} + 256c_4xe^{4x}
\end{aligned}$$

into $y^{(4)} - 8y^{(3)} + 16y''$ an obtain

$$\begin{cases} \Big((256c_3 + 256c_4)e^{4x} + 256c_4xe^{4x}\Big) \\ -8\Big((64c_3 + 48c_4)e^{4x} + 64c_4xe^{4x}\Big) \\ +16\Big((16c_3 + 8c_4)e^{4x} + 16c_4xe^{4x}\Big) \end{cases}$$

$$= \Big(256-8(64)+16(16)\Big)c_3e^{4x} + \Big(256-8(48)+16(8)\Big)c_4e^{4x} + \Big(256-8(64)+16(16)\Big)c_4xe^{4x}$$

Notice that

$$256 - 8(64) + 16(16) = 2^8 - 2^3(2^6) + 2^4(2^4) = 2^8(1 - 2 + 1) = 0$$

and

$$256 - 8(48) + 16(8) = 2^8 - (2^3)(3)(2^4) + 2^4(2^3) = 2^7(2 - 3 + 1) = 0.$$

Our proposed answer works.