Problem 16 in Section 3.2. The problem tells us that $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = xe^{2x}$ all are solutions of the Differential equation $y''' - 5y'' + 8y' - 4y = 0$. We are supposed to solve the Initial Value Problem

$$y''' - 5y'' + 8y' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 4, \quad y''(0) = 0.$$

Solution. The general solution of $y''' - 5y'' + 8y' - 4y = 0$ is

$$y = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}.$$

Our job is to evaluate the constants. We calculate

$$\begin{aligned} y' =& c_1 e^x + 2c_2 e^{2x} + c_3 e^{2x} + 2c_3 x e^{2x} \\ =& c_1 e^x + (2c_2 + c_3)e^{2x} + 2c_3 x e^{2x} \\ y'' =& c_1 e^x + 2(2c_2 + c_3)e^{2x} + 2c_3 e^{2x} + 4c_3 x e^{2x} \\ =& c_1 e^x + (4c_2 + 4c_3)e^{2x} + 4c_3 x e^{2x} \end{aligned}$$

We must solve

$$\begin{aligned} 1 =& c_1 + c_2 \\ 4 =& c_1 + 2c_2 + c_3 \\ 0 =& c_1 + 4c_2 + 4c_3 \end{aligned}$$

Replace Equation 2 by Equation 2 minus Equation 1.
Replace Equation 3 by Equation 3 minus Equation 1.

$$\begin{cases} 1 = c_1 + c_2 & \\ 3 = \qquad c_2 + \ \ c_3 & \\ -1 = \qquad 3c_2 + 4c_3 & \end{cases}$$

Replace Equation 3 by Equation 3 minus 3 times Equation 2.

$$\begin{cases} 1 = c_1 + c_2 & \\ 3 = \qquad c_2 + c_3 & \\ -10 = \qquad\qquad c_3 & \end{cases}$$

Thus, $c_3 = -10$, $c_2 = 13$, and $c_1 = -12$.

The solution of the Initial Value Problem is

$$\boxed{y = -12e^x + 13e^{2x} - 10xe^{2x}.}$$

Check. We compute

$$\begin{aligned}
y =& -12e^{x} + 13e^{2x} - 10xe^{2x} \\
y' =& -12e^{x} + 26e^{2x} - 10e^{2x} - 20xe^{2x} \\
=& -12e^{x} + 16e^{2x} - 20xe^{2x} \\
y'' =& -12e^{x} + 32e^{2x} - 20e^{2x} - 40xe^{2x} \\
=& -12e^{x} + 12e^{2x} - 40xe^{2x} \\
y''' =& -12e^{x} + 24e^{2x} - 40e^{2x} - 80xe^{2x} \\
=& -12e^{x} - 16e^{2x} - 80xe^{2x}
\end{aligned}$$

We plug y, y', y'', and y''' into $y''' - 5y'' + 8y' - 4y$ and obtain

$$\begin{cases}
-12e^{x} - 16e^{2x} - 80xe^{2x} \\
-5(-12e^{x} + 12e^{2x} - 40xe^{2x}) \\
+8(-12e^{x} + 16e^{2x} - 20xe^{2x}) \\
-4(-12e^{x} + 13e^{2x} - 10xe^{2x})
\end{cases}$$

$= e^{x}(-12+60-96+48)+e^{2x}(-16-60+128-52)+xe^{2x}(-80+200-160+40)$

$= 0.$✓

Also we compute $y(0) = -12e^{0} + 13e^{0} - 10(0) = 1$✓, $y'(0) = -12e^{0} + 16e^{0} - 20(0) = 4$✓, and $y''(0) = -12e^{0} + 12e^{0} - 40(0) = 0$✓.

The proposed solution does everything it is supposed to do. It is correct.